

# THE MIXED BOUNDARY VALUE PROBLEM, KREIN RESOLVENT FORMULAS AND SPECTRAL ASYMPTOTIC ESTIMATES

GERD GRUBB

Department of Mathematical Sciences, Copenhagen University, Universitetsparken  
5, DK-2100 Copenhagen, Denmark. E-mail [grubb@math.ku.dk](mailto:grubb@math.ku.dk)

**ABSTRACT.** For a second-order symmetric strongly elliptic operator  $A$  on a smooth bounded open set in  $\mathbb{R}^n$ , the mixed problem is defined by a Neumann-type condition on a part  $\Sigma_+$  of the boundary and a Dirichlet condition on the other part  $\Sigma_-$ . We show a Krein resolvent formula, where the difference between its resolvent and the Dirichlet resolvent is expressed in terms of operators acting on Sobolev spaces over  $\Sigma_+$ . This is used to obtain a new Weyl-type spectral asymptotics formula for the resolvent difference (where upper estimates were known before), namely  $s_j j^{2/(n-1)} \rightarrow C_{0,+}^{2/(n-1)}$ , where  $C_{0,+}$  is proportional to the area of  $\Sigma_+$ , in the case where  $A$  is principally equal to the Laplacian.

The mixed boundary value problem for a second-order strongly elliptic symmetric operator  $A$  on a smooth bounded open set  $\Omega \subset \mathbb{R}^n$  with boundary  $\Sigma$ , in case of the Laplacian also called the Zaremba problem, is defined by a Neumann-type condition on a part of the boundary  $\Sigma_+$  and a Dirichlet condition on the other part  $\Sigma_-$ . It does not have the regularity of standard elliptic boundary problems (the  $L_2$ -domain is at best in  $H^{\frac{3}{2}-\varepsilon}(\Omega)$ ). It has been analyzed with regards to regularity and mapping properties e.g. in Peetre [P61, P63], Shamir [S68], Eskin [E81], Pryde [P81], Rempel and Schulze [RS83], Simanca [S87], Harutyunyan and Schulze [HS08].

We shall here study it from the point of view of extension theory for elliptic operators. There has been a recent revival in the interest for connections between abstract extension theories for operators in Hilbert space (as initiated by Krein [K47], Vishik [V52], Birman [B62], Grubb [G68] and others) and interpretations to boundary value problems for partial differential operators. Cf. e.g. Amrein and Pearson [AP04], Pankrashkin [P06], Behrndt and Langer [BL07], Ryzhov [R07], Brown, Marletta, Naboko and Wood [BMNW08], Alpay and Behrndt [AB09], Malamud [M10], based on boundary triples theory (as developed from the book of Gorbachuk and Gorbachuk [GG91] and its sources). Other methods are used in the works of Brown, Grubb and Wood [BGW09], [G08], Posilicano and Raimondi [PR09], Gesztesy and Mitrea [GM08, GM09, GM11] (and their references); see also Grubb [G11, G11a, G11b] and Abels, Grubb and Wood [AGW11]. One of the interesting aims has been to derive Krein resolvent formulas that link the resolvent of a general operator with the resolvent of a fixed reference operator by expressing the difference in terms of operators connected to the boundary.

For the mixed problem, a Krein resolvent formula connecting the operator to the Dirichlet realization was worked out in [P06], based on boundary triples theory. A different formula results from [G68, G74], see also [BGW09], Sect. 3.2.5. Observations on the connection with the Neumann realization were given in [M10]. An upper bound for the spectral behavior of the resolvent difference was shown by Birman in [B62].

In the present paper we shall work out in detail several Krein resolvent formulas for the mixed problem. The primary result is a formula where the difference between the resolvents for the mixed

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problem and the Dirichlet problem is expressed explicitly in terms of operators acting over the subset  $\Sigma_+$ ; this is based on the universal description from [G68] in terms of operators between closed subspaces of the nullspace of the maximal operator. In addition, we show some other explicit formulas related to those of [P06]. Mixed problems for  $-\Delta$  on creased domains are briefly considered, and we establish a Krein formula for quasi-convex Lipschitz domains as defined in [GM11].

As an application of our primary formula in the smooth case, we show how it leads to a new result giving a Weyl-type spectral asymptotic estimate for the resolvent difference, with the constant defined by an integral over  $\Sigma_+$ ; this sharpens considerably the upper estimates known earlier. The proof draws on various results for nonstandard pseudodifferential operators on  $\Sigma$ .

## 1. INTRODUCTION

On a bounded smooth open subset  $\Omega$  of  $\mathbb{R}^n$  with boundary  $\partial\Omega = \Sigma$ , consider a second-order symmetric differential operator with real coefficients in  $C^\infty(\overline{\Omega})$  and an associated sesquilinear form

$$(1.1) \quad Au = -\sum_{j,k=1}^n \partial_j(a_{jk}(x)\partial_k u) + a_0(x)u,$$

$$(1.2) \quad a(u, v) = \sum_{j,k=1}^n (a_{jk}\partial_k u, \partial_j v) + (a_0 u, v).$$

$A$  is assumed strongly elliptic, i.e.,  $\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \geq c_0|\xi|^2$  for  $x \in \Omega$ ,  $\xi \in \mathbb{R}^n$ , with  $c_0 > 0$ .

Denote  $u|_\Sigma = \gamma_0 u$ , and  $\sum_j n_j \gamma_0(\partial_j u) = \gamma_1 u$ , where  $\vec{n} = (n_1, \dots, n_n)$  is the interior unit normal to the boundary. Introduce the conormal derivative  $\nu$  and a variant  $\chi$  (Neumann-type boundary operators)

$$(1.3) \quad \nu u = \sum_{j,k=1}^n n_j \gamma_0(a_{jk}\partial_k u), \quad \chi u = \nu u - b\gamma_0 u.$$

$\nu$  enters in the “halfways Green’s formula” (for sufficiently smooth functions)

$$(1.4) \quad (Au, v)_{L_2(\Omega)} - a(u, v) = (\nu u, \gamma_0 v)_{L_2(\Sigma)}.$$

Consider the realizations  $A_\gamma$ ,  $A_\nu$ ,  $A_\chi$  resp.  $A_{\chi, \Sigma_+}$  of  $A$  defined via sesquilinear forms to represent the respective boundary conditions

$$(1.5) \quad \begin{aligned} \gamma_0 u &= 0 \text{ on } \Sigma, \text{ the Dirichlet condition,} \\ \nu u &= 0 \text{ on } \Sigma, \text{ the Neumann condition,} \\ \chi u &= 0 \text{ on } \Sigma, \text{ a Robin (Neumann-type) condition,} \\ \chi u &= 0 \text{ on } \Sigma_+, \gamma_0 u = 0 \text{ on } \Sigma \setminus \Sigma_+, \text{ a mixed condition;} \end{aligned}$$

here  $b$  is a bounded measurable real function and  $\Sigma_+$  is a closed subset of  $\Sigma$ . These realizations are selfadjoint, and by addition of a large constant to  $a_0$  we can obtain that they have positive lower bounds. Their resolvents are compact operators. Note that  $A_\chi$  equals  $A_\nu$  for  $b = 0$ .

For a compact operator  $B$  in a Hilbert space  $H$ ,  $s_j(B)$  denotes the  $j$ -th eigenvalue of  $(B^*B)^{\frac{1}{2}}$  (the  $j$ -th s-number or singular value of  $B$ ), counted with multiplicities.

Birman showed in [B62]:

$$(1.6) \quad s_j(A_\chi^{-1} - A_\gamma^{-1}) \text{ and } s_j(A_{\chi, \Sigma_+}^{-1} - A_\gamma^{-1}) \text{ are } O(j^{-2/(n-1)}) \text{ for } j \rightarrow \infty;$$

also valid for exterior domains. The estimate for  $A_\chi^{-1} - A_\gamma^{-1}$  was later improved to an asymptotic estimate (in [G74] and [BS80], the latter including exterior domains):

$$(1.7) \quad \lim_{j \rightarrow \infty} s_j(A_\chi^{-1} - A_\gamma^{-1}) j^{2/(n-1)} = C_0^{2/(n-1)},$$

for smooth  $b$ , where

$$(1.8) \quad C_0 = \frac{1}{(n-1)(2\pi)^{n-1}} \int_{\Sigma} \int_{|\xi'|=1} (\|\tilde{k}^0\|_{L_2(\mathbb{R}_+)} |p^0|^{1/2})^{n-1} d\omega(\xi') dx';$$

this has been extended to nonsmooth  $b$  in [G11a] (the ingredients in the formula are explained around Th. 2.4 there). For the difference with  $A_{\chi, \Sigma_+}^{-1}$  an asymptotic estimate does not seem to have been obtained before; it is one of the aims of the present paper.

In Section 2, we briefly recall some elements of the old extension theory from [G68, G74]. In Section 3, we show how the method of Birman [B62] can be used in combination with later estimates to make a small improvement of his result for mixed problems, valid for nonsmooth  $b$  and  $\Sigma_+$ .

In Section 4, we analyze the structure of  $A_{\chi, \Sigma_+}$  in terms of the characterization from [G68] in more detail, describing the operator  $L^\lambda: X \rightarrow X^*$  that  $A_{\chi, \Sigma_+} - \lambda$  corresponds to when  $\lambda \in \varrho(A_\gamma)$  (the resolvent set):

**Theorem A.** *When  $b$  and the subset  $\Sigma_+$  are smooth, then  $X = H_0^{-\frac{1}{2}}(\Sigma_+)$ , and  $L^\lambda$  acts like minus the Dirichlet-to-Neumann pseudodifferential operator truncated to  $\Sigma_+$ ,  $-P_{\gamma, \chi, +}^\lambda = -r^+ \chi K_\gamma^\lambda e^+$ , with domain  $D(L^\lambda) \subset H_0^{1-\varepsilon}(\Sigma_+)$  (any  $\varepsilon > 0$ ); here  $K_\gamma^\lambda$  is the Poisson operator for the Dirichlet problem for  $A - \lambda$ .*

*For  $\lambda \in \varrho(A_{\chi, \Sigma_+}) \cap \varrho(A_\gamma)$  there is a Kreĭn resolvent formula:*

$$(1.9) \quad (A_{\chi, \Sigma_+} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1} = -K_{\gamma, \chi}^\lambda (P_{\gamma, \chi, +}^\lambda)^{-1} (K_{\gamma, \chi}^{\bar{\lambda}})^*.$$

Several other Kreĭn-type formulas are shown involving the Poisson operators for the Dirichlet or Neumann problems.

In Section 5, we restrict the attention to operators principally like the Laplacian. Here we use methods for nonstandard pseudodifferential operators to deduce from (1.9):

**Theorem B.** *When  $A = -\Delta + a_0(x)$ , then for any  $\lambda \in \varrho(A_{\chi, \Sigma_+}) \cap \varrho(A_\gamma)$ ,*

$$(1.10) \quad \lim_{j \rightarrow \infty} s_j ((A_{\chi, \Sigma_+} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1}) j^{2/(n-1)} = C_{0, +}^{2/(n-1)},$$

where  $C_{0, +}$  is a constant proportional to the area of  $\Sigma_+$ ;

$$(1.11) \quad C_{0, +} = \frac{1}{(n-1)(2\pi)^{n-1}} \int_{\Sigma_+} \int_{|\xi'|=1} (\|\tilde{k}^0\|_{L_2(\mathbb{R}_+)} |p^0|^{1/2})^{n-1} d\omega(\xi') dx'.$$

Remark 3.3 and Section 4.3 give informations on cases where  $\Omega$  is not smooth.

A general technique for extending the estimates to exterior domains can be found in [G11].

## 2. PRELIMINARIES

### 2.1 Definition of the operators.

The spaces  $H^s(\Omega)$ ,  $H^s(\Sigma)$  are the standard Sobolev spaces, with the norm denoted  $\|u\|_s$ ;  $H_0^s(\Omega)$  (or  $H_0^s(\overline{\Omega})$ ) stands for the space of distributions in  $H^s(\mathbb{R}^n)$  with support in  $\overline{\Omega}$ . We use the notation  $(\cdot, \cdot)_{-s, s}$  for the sesquilinear duality between  $H^{-s}(\Sigma)$  and  $H^s(\Sigma)$ ,  $s \in \mathbb{R}$ ; it reduces to the  $L_2$ -scalar product when applied to functions in  $L_2(\Sigma)$ .

It is known e.g. from Lions and Magenes [LM68] that  $\gamma_0$  resp.  $\gamma_1, \nu$  extend to continuous mappings from  $H^s(\Omega) \cap D(A_{\max})$  to  $H^{s-\frac{1}{2}}(\Sigma)$  resp.  $H^{s-\frac{3}{2}}(\Sigma)$ , any  $s \geq 0$ , allowing extensions of Green's formulas. In particular, for  $u \in H^1(\Omega) \cap D(A_{\max})$ ,  $v \in H^1(\Omega)$ , (1.4) holds with the scalar product in  $L_2(\Sigma)$  replaced by the sesquilinear duality between  $H^{-\frac{1}{2}}(\Sigma)$  and  $H^{\frac{1}{2}}(\Sigma)$ .

The realizations of  $A$  are the linear operators  $\tilde{A}$  satisfying  $A_{\min} \subset \tilde{A} \subset A_{\max}$ , where  $A_{\min}$  and  $A_{\max}$  act like  $A$  with domains  $D(A_{\min}) = H_0^2(\Omega)$  resp.  $D(A_{\max}) = \{u \in L_2(\Omega) \mid Au \in L_2(\Omega)\}$ ;  $A_{\min}$  is the closure of  $A|_{C_0^\infty}$ , and  $A_{\max} = A_{\min}^*$ .

Our assumptions imply that

$$(2.1) \quad a(u, u) \geq c\|u\|_{H^1(\Omega)}^2 - k\|u\|_{L_2(\Omega)}^2 \text{ for } u \in H^1(\Omega),$$

with  $c > 0$ ,  $k \geq 0$ . Then the realizations  $A_\gamma$ , etc., can all be defined via variational constructions from sesquilinear forms, namely:

$$(2.2) \quad \begin{aligned} a_\gamma(u, v) &= a(u, v) \text{ on } D(a_\gamma) = H_0^1(\Omega) \text{ leads to } A_\gamma, \\ a_\nu(u, v) &= a(u, v) \text{ on } D(a_\nu) = H^1(\Omega) \text{ leads to } A_\nu, \\ a_\chi(u, v) &= a(u, v) + (b\gamma_0 u, \gamma_0 v)_{L_2(\Sigma)} \text{ on } D(a_\chi) = H^1(\Omega) \text{ leads to } A_\chi, \\ a_{\chi, \Sigma_+}(u, v) &= a(u, v) + (b\gamma_0 u, \gamma_0 v)_{L_2(\Sigma_+)} \text{ on } D(a_{\chi, \Sigma_+}) = H_{\Sigma_+}^1(\Omega) \text{ leads to } A_{\chi, \Sigma_+}; \end{aligned}$$

here

$$(2.3) \quad H_{\Sigma_+}^1(\Omega) = \{u \in H^1(\Omega) \mid \text{supp } \gamma_0 u \subset \Sigma_+\}.$$

The last case (that covers the two preceding cases when  $\Sigma_+ = \Sigma$  or  $b = 0$ ) is explained below. Since  $\|\gamma_0 u\|_{L_2(\Sigma)}^2 \leq c'\|u\|_{\frac{3}{4}}^2 \leq \varepsilon\|u\|_1^2 + C(\varepsilon)\|u\|_0^2$  for any  $\varepsilon$ , we infer from (2.1) that when  $K$  is a constant  $\geq \text{ess sup } |b(x)|$ ,  $a_\chi(u, u) \geq a(u, u) - K\|\gamma_0 u\|_0^2$ , and hence

$$a_\chi(u, u) \geq c_1\|u\|_1^2 - k_1\|u\|_0^2, \text{ for } u \in H^1(\Omega),$$

where  $c_1 < c$  is close to  $c$  and  $k_1 \geq k$  is a large constant. Then each of the sesquilinear forms in (2.2) satisfies such an inequality on its domain. Defining  $\chi_K = \nu + K\gamma_0$  (the case  $b = -K$ ), we also have that  $a_{\chi_K}(u, v) = a(u, v) - K(\gamma_0 u, \gamma_0 v)_\Sigma$  satisfies such an inequality. We can (after a fixed choice of the constant  $K$ ) replace  $A$  by  $A + k_1$ , i.e. add the constant  $k_1$  to the coefficient  $a_0$  in (2.1); then all the resulting sesquilinear forms, including  $a_{\chi_K}$ , are positive. *For simplicity,  $A + k_1$  and  $a(u, v) + k_1 \cdot (u, v)$  will in the following again be denoted  $A$  and  $a(u, v)$ .*

We now recall the construction of  $A_{\chi, \Sigma_+}$ . The sesquilinear form  $a_{\chi, \Sigma_+}$  on  $V = H_{\Sigma_+}^1(\Omega)$  in  $H = L_2(\Omega)$  defines an operator  $A_{\chi, \Sigma_+}$  by

$$(2.4) \quad \begin{aligned} D(A_{\chi, \Sigma_+}) &= \{u \in V \mid \exists f \in H \text{ such that } a_{\chi, \Sigma_+}(u, v) = (f, v) \text{ for all } v \in V\}, \\ A_{\chi, \Sigma_+} u &= f. \end{aligned}$$

By J.L. Lions' version of the Lax-Milgram lemma, as recalled e.g. in [G09], Sect. 12.4, this defines a selfadjoint operator with the same lower bound as  $a_{\chi, \Sigma_+}$ . Clearly,  $A_{\chi, \Sigma_+}$  extends  $A|_{C_0^\infty}$ , hence  $A_{\min}$ , and in view of the selfadjointness is a restriction of  $A_{\min}^* = A_{\max}$ , so it is a realization of  $A$ . By (1.4),

$$(2.5) \quad (Au, v) - a_{\chi, \Sigma_+}(u, v) = (\nu u, \gamma_0 v)_{-\frac{1}{2}, \frac{1}{2}} - (b\gamma_0 u, \gamma_0 v)_{L_2(\Sigma)} = (\chi u, \gamma_0 v)_{-\frac{1}{2}, \frac{1}{2}},$$

when  $v \in H_{\Sigma_+}^1(\Omega)$ . Thus, when  $u \in D(A_{\max}) \cap H_{\Sigma_+}^1(\Omega)$ ,  $a_{\chi, \Sigma_+}(u, v) = (Au, v)$  holds for all  $v \in H_{\Sigma_+}^1(\Omega)$  precisely when the distribution  $\chi u$  vanishes on the  $H^{\frac{1}{2}}$ -functions supported in  $\Sigma_+$ . In this sense,  $A_{\chi, \Sigma_+}$  represents the boundary condition  $\gamma_0 u = 0$  on  $\Sigma \setminus \Sigma_+$ ,  $\chi u = 0$  on  $\Sigma_+$ .

The boundary condition can be made more explicit when  $\Sigma_+$  is a smooth subset of  $\Sigma$ . We then set  $\Sigma_- = \Sigma \setminus \Sigma_+$ , and have that  $\Sigma = \Sigma_+ \cup \Sigma_-$ , with  $\Sigma_+^\circ \cup \Sigma_-^\circ$  dense in  $\Sigma$ . Then for  $s \in \mathbb{R}$ , we denote by  $H_0^s(\Sigma_+)$  the closed subspace of  $H^s(\Sigma)$  consisting of the elements with support in  $\Sigma_+$ . Here  $C_0^\infty(\Sigma_+^\circ)$  is a dense subspace, and it should be noted that for  $s + \frac{1}{2} \in \mathbb{N}$ , the space is different from the space obtained by closure of  $C_0^\infty(\Sigma_+^\circ)$  in  $H^s(\Sigma_+^\circ)$ . For  $s \in \mathbb{R}$ , the latter space  $H^s(\Sigma_+^\circ)$  consists of the restrictions to  $\Sigma_+^\circ$  of distributions in  $H^s(\Sigma)$ , provided with the quotient norm. The spaces  $H_0^s(\Sigma_+)$  and  $H^{-s}(\Sigma_+^\circ)$  are dual with respect to an extension of the  $L_2$  scalar product, for all  $s \in \mathbb{R}$ .

**Lemma 2.1.** *When  $\Sigma_+$  is smooth,*

$$(2.6) \quad D(A_{\chi, \Sigma_+}) = \{u \in H^1(\Omega) \cap D(A_{\max}) \mid \gamma_0 u \in H_0^{\frac{1}{2}}(\Sigma_+), \chi u = 0 \text{ on } \Sigma_+^{\circ}\}.$$

*Proof.* Note first that  $\gamma_0 H_{\Sigma_+}^1(\Omega) = H_0^{\frac{1}{2}}(\Sigma_+)$ , since  $\gamma_0 H^1(\Omega) = H^{\frac{1}{2}}(\Sigma)$  and  $H_0^{\frac{1}{2}}(\Sigma_+)$  is the subspace of  $H^{\frac{1}{2}}(\Sigma)$  consisting of the functions supported in  $\Sigma_+$ . Moreover,  $C_0^\infty(\Sigma_+^{\circ})$  is dense in  $H_0^{\frac{1}{2}}(\Sigma_+)$  and is the image by  $\gamma_0$  of the space of  $C^\infty(\overline{\Omega})$ -functions  $\psi$  with  $\gamma_0 \psi$  supported in  $\Sigma_+^{\circ}$ .

When  $u$  is in the right-hand side of (2.6), then

$$\langle \chi u, \gamma_0 \psi \rangle = 0 \text{ for } \gamma_0 \psi \in C_0^\infty(\Sigma_+^{\circ});$$

hence by the denseness of  $C_0^\infty(\Sigma_+^{\circ})$  in  $H_0^{\frac{1}{2}}(\Sigma_+)$ ,

$$(\chi u, \gamma_0 v)_{-\frac{1}{2}, \frac{1}{2}} = 0 \text{ for } v \in H_{\Sigma_+}^1(\Omega);$$

so  $u \in D(A_{\chi, \Sigma_+})$ . Conversely, if  $u \in D(A_{\chi, \Sigma_+})$ , then  $u \in D(A_{\max}) \cap H_{\Sigma_+}^1(\Omega)$  implies  $\gamma_0 u \in H_0^{\frac{1}{2}}(\Sigma_+)$ , and since  $\chi u$  vanishes on  $H^{\frac{1}{2}}$ -functions supported in  $\Sigma_+$ , it vanishes in particular on  $C_0^\infty(\Sigma_+^{\circ})$ , i.e.,  $\nu u - b\gamma_0 u = 0$  on  $\Sigma_+^{\circ}$ .  $\square$

## 2.2 Abstract extension theories.

We shall now connect the operators with the theory of Kreĭn [K47], Vishik [V52], Birman [B56], Grubb [G68, G70] (the latter also recalled in [BGW09], the abstract part in [G09], Ch. 13). The theory of [G68] extends and completes that of [V52] by giving a universal description of all adjoint pairs of extensions of a dual pair of injective operators. We here just briefly recall how it describes the extensions  $\tilde{A}$  of a symmetric positive operator  $A_{\min}$  with  $A_{\min} \subset \tilde{A} \subset A_{\max} = A_{\min}^*$ .

The operators act in a Hilbert space  $H$  (in the concrete application,  $H = L_2(\Omega)$ ). Let  $A_\gamma$  be the Friedrichs extension of  $A_{\min}$  (in the application it will be the Dirichlet realization), and let  $Z = \ker A_{\max}$ . Define the decomposition

$$(2.7) \quad D(A_{\max}) = D(A_\gamma) \dot{+} Z, \text{ with notation } u = u_\gamma + u_\zeta,$$

where  $u_\gamma = \text{pr}_\gamma u = A_\gamma^{-1} A_{\max} u$ ,  $u_\zeta = u - u_\gamma = (1 - \text{pr}_\gamma)u = \text{pr}_\zeta u$ . This is used in [G68] to show that there is a 1–1 correspondence between the closed realizations  $\tilde{A}$  of  $A$  and the closed, densely defined operators between closed subspaces of  $Z$ :

$$(2.8) \quad \tilde{A} \text{ closed} \longleftrightarrow \begin{cases} V, W \subset Z, \text{ closed subspaces,} \\ T: V \rightarrow W \text{ closed, densely defined,} \end{cases}$$

where  $D(T) = \text{pr}_\zeta D(\tilde{A})$ ,  $X = \overline{D(T)}$ ,  $W = \overline{\text{pr}_\zeta D(\tilde{A}^*)}$ , and  $Tu_\zeta = \text{pr}_W(A_{\max} u)$  (here  $\text{pr}_W$  denotes orthogonal projection onto  $W$ ). The operator  $\tilde{A}^*$  corresponds similarly to  $T^*: W \rightarrow V$ , and many properties carry over between  $\tilde{A}$  and  $T$ . For example,  $\tilde{A}$  is invertible (i.e. bijective) if and only if  $T$  is so, and then we have an abstract resolvent formula:

$$(2.9) \quad \tilde{A}^{-1} = A_\gamma^{-1} + i_V T^{-1} \text{pr}_W,$$

where  $i_V$  denotes the injection  $V \hookrightarrow H$ .

In particular,  $\tilde{A}$  is selfadjoint if and only if:  $V = W$  and  $T: V \rightarrow V$  is selfadjoint. Then in the invertible case,

$$(2.10) \quad \tilde{A}^{-1} = A_\gamma^{-1} + i_V T^{-1} \text{pr}_V.$$

Positivity of  $\tilde{A}$  holds if and only if  $T$  is positive.

For the positive selfadjoint operators, there is also a connection between the associated sesquilinear forms. (When  $S$  is a positive selfadjoint operator in a Hilbert space  $H$ , the associated sesquilinear form  $s$  has as its domain  $D(s)$  the completion of  $D(S)$  in the norm  $(Su, u)^{\frac{1}{2}}$ , stronger than the  $H$ -norm; here  $D(s) \subset H$ , and the form  $s(u, v)$  is the extension by continuity of  $(Su, v)$  to  $D(s)$ . Then  $S$  is defined from  $s$  by the Lax-Milgram construction.) When  $\tilde{A}$  is positive selfadjoint, corresponding to the positive selfadjoint operator  $T$  in  $V$ , the associated sesquilinear form  $\tilde{a}$  can be written

$$(2.11) \quad \tilde{a}(u, v) = a_\gamma(u_\gamma, v_\gamma) + t(u_\zeta, v_\zeta) \text{ on } D(\tilde{a}) = D(a_\gamma) \dot{+} D(t),$$

where  $t$  on  $D(t) \subset V$  is the sesquilinear form associated with  $T$ ; the decomposition  $u = u_\gamma + u_\zeta$  used here is a continuous extension to  $D(a_\gamma) \dot{+} Z$  of the decomposition (2.7) above.

The description of selfadjoint extensions in terms of sesquilinear forms is already found in [K47] and [B56]; [G70] moreover treats nonselfadjoint extensions.

Much of the theory holds unchanged if we replace the “reference operator”  $A_\gamma$  by another selfadjoint positive realization of  $A$ , say  $A_\nu$  (which will in the application be taken as the Neumann realization  $A_\nu$ ). There is again a decomposition

$$D(A_{\max}) = D(A_\nu) \dot{+} Z, \text{ say with notation } u = u_\nu + u_{\zeta,1},$$

where  $u_\nu = \text{pr}_\nu u = A_\nu^{-1} A_{\max} u$ ,  $u_{\zeta,1} = u - u_\nu = (1 - \text{pr}_\nu)u = \text{pr}_{\zeta,1} u$ , and there is a 1–1 correspondence

$$(2.12) \quad \tilde{A} \text{ closed} \longleftrightarrow \begin{cases} V_1, W_1 \subset Z, \text{ closed subspaces,} \\ T_1: V_1 \rightarrow W_1 \text{ closed, densely defined,} \end{cases}$$

where  $D(T_1) = \text{pr}_{\zeta,1} D(\tilde{A})$ ,  $X_1 = \overline{D(T_1)}$ ,  $W_1 = \overline{\text{pr}_{\zeta,1} D(\tilde{A}^*)}$ , and  $T_1 u_{\zeta,1} = \text{pr}_{W_1}(A_{\max} u)$ ; again  $\tilde{A}$  is selfadjoint or invertible if and only if  $T_1$  is so, and in the invertible case,

$$(2.13) \quad \tilde{A}^{-1} = A_\nu^{-1} + \text{i}_{V_1} T_1^{-1} \text{pr}_{W_1}.$$

However, positivity does not in general carry over between  $\tilde{A}$  and  $T_1$ , and the information on associated sesquilinear forms does not generalize to this situation, since those facts depended on  $A_\gamma$  being the Friedrichs extension of  $A_{\min}$ .

### 2.3 Concrete boundary conditions. Dirichlet reference operator.

We now explain the interpretation to concrete boundary conditions worked out in [G68, G74]. Along with (1.4) we have the full Green’s formula

$$(2.14) \quad (Au, v)_{L_2(\Omega)} - (u, Av)_{L_2(\Omega)} = (\nu u, \gamma_0 v)_{L_2(\Sigma)} - (\gamma_0 u, \nu v)_{L_2(\Sigma)}, \text{ for } u, v \in H^2(\Omega);$$

it extends e.g. to  $u \in D(A_{\max})$ ,  $v \in H^2(\Omega)$ , with the  $L_2(\Sigma)$ -scalar products replaced by suitable Sobolev space dualities, but it cannot be extended to  $u, v \in D(A_{\max})$ .

Denote by  $K_\gamma$  resp.  $K_\nu$  the Poisson operator solving the Dirichlet problem resp. Neumann problem

$$Au = 0 \text{ in } \Omega, \text{ with } \gamma_0 u = \varphi, \text{ resp. } \nu u = \psi;$$

they have the mapping properties

$$K_\gamma: H^{s-\frac{1}{2}}(\Sigma) \rightarrow H^s(\Omega), \quad K_\nu: H^{s-\frac{3}{2}}(\Sigma) \rightarrow H^s(\Omega), \text{ for all } s \in \mathbb{R}.$$

In particular,  $\gamma_0$  and  $\nu$  define *homeomorphisms* of  $Z$  onto  $H^{-\frac{1}{2}}(\Sigma)$  resp.  $H^{-\frac{3}{2}}(\Sigma)$ , with  $K_\gamma$  resp.  $K_\nu$  acting as inverses.

Let  $\tilde{A}$  correspond to  $T: V \rightarrow W$  as in (2.8). Let  $X = \gamma_0(V)$ ,  $Y = \gamma_0(W)$ , closed subspaces of  $H^{-\frac{1}{2}}(\Sigma)$ , and introduce the notation for the connecting homeomorphisms

$$(2.15) \quad \gamma_V: V \xrightarrow{\sim} X, \quad \gamma_W: W \xrightarrow{\sim} Y.$$

By use of these homeomorphisms,  $T: V \rightarrow W$  is carried over to a map  $L: X \rightarrow Y^*$ :

$$\begin{array}{ccc} V & \xrightarrow[\gamma_V]{\sim} & X \\ T \downarrow & & \downarrow L \\ W & \xleftarrow[\gamma_W^*]{\sim} & Y^* \end{array} \quad D(L) = \gamma_0 D(T) = \gamma_0 D(\tilde{A}).$$

In other words,

$$L = (\gamma_W^*)^{-1} T \gamma_V^{-1}.$$

In the case where  $\tilde{A}$  is invertible, the abstract resolvent formula (2.9) carries over to the formula:

$$(2.16) \quad \tilde{A}^{-1} = A_\gamma^{-1} + K_{\gamma,X} L^{-1} (K_{\gamma,Y})^*$$

where

$$(2.17) \quad K_{\gamma,X} = i_V \gamma_V^{-1}: X \rightarrow V \subset H, \quad (K_{\gamma,Y})^* = (\gamma_W^*)^{-1} \text{pr}_W: H \rightarrow Y^*;$$

(2.16) is a *Kreĭn resolvent formula*. In particular, if  $V = W = Z$ , then  $X = Y = H^{-\frac{1}{2}}(\Sigma)$ , and (2.16) takes the form

$$(2.18) \quad \tilde{A}^{-1} = A_\gamma^{-1} + K_\gamma L^{-1} K_\gamma^*,$$

where  $L$  goes from  $D(L) \subset H^{-\frac{1}{2}}(\Sigma)$  to  $H^{\frac{1}{2}}(\Sigma)$ .

To see how  $L$  enters in a concrete boundary condition for  $\tilde{A}$  we define some additional operators, namely the *Dirichlet-to-Neumann* and *Neumann-to-Dirichlet* pseudodifferential operators ( $\psi$ do's)  $P_{\gamma,\nu}$  and  $P_{\nu,\gamma}$ , and the associated *reduced trace operators*  $\Gamma_\nu$  and  $\Gamma_\gamma$ :

$$(2.19) \quad \begin{aligned} P_{\gamma,\nu} &= \nu K_\gamma, \text{ } \psi\text{do of order } 1, \quad \Gamma_\nu = \nu - P_{\gamma,\nu} \gamma_0: D(A_{\max}) \rightarrow H^{\frac{1}{2}}(\Sigma); \\ P_{\nu,\gamma} &= \gamma_0 K_\nu, \text{ } \psi\text{do of order } -1, \quad \Gamma_\gamma = \gamma_0 - P_{\nu,\gamma} \nu: D(A_{\max}) \rightarrow H^{\frac{3}{2}}(\Sigma). \end{aligned}$$

(We here use the notation of the pseudodifferential boundary operator calculus, initiated by Boutet de Monvel [B71] and further developed in [G84, G96], see also [G09].) More generally,  $P_{\beta,\beta'}$  denotes the mapping from  $\beta u$  to  $\beta' u$ , when  $u \in Z$  is uniquely determined from  $\beta u$ .

The reduced trace operators are used to establish generalized Green's formulas valid for  $u, v \in D(A_{\max})$ :

$$(2.20) \quad \begin{aligned} (Au, v)_{L_2(\Omega)} - (u, Av)_{L_2(\Omega)} &= (\Gamma_\nu u, \gamma_0 v)_{\frac{1}{2}, -\frac{1}{2}} - (\gamma_0 u, \Gamma_\nu v)_{-\frac{1}{2}, \frac{1}{2}}, \\ (Au, v)_{L_2(\Omega)} - (u, Av)_{L_2(\Omega)} &= (\nu u, \Gamma_\gamma v)_{-\frac{3}{2}, \frac{3}{2}} - (\Gamma_\gamma u, \nu v)_{\frac{3}{2}, -\frac{3}{2}}. \end{aligned}$$

One can then show:

$D(\tilde{A})$  consists of the functions  $u \in D(A_{\max})$  that satisfy:

$$(2.21) \quad \gamma_0 u \in D(L), \quad (\Gamma_\nu u, \varphi)_{\frac{1}{2}, -\frac{1}{2}} = (L \gamma_0 u, \varphi)_{Y^*, Y} \text{ for all } \varphi \in Y.$$

The second condition may be rewritten as  $i_Y^* \Gamma_\nu u = L \gamma_0 u$ , where  $i_Y^*: H^{\frac{1}{2}}(\Sigma) \rightarrow Y^*$  is the adjoint of  $i_Y: Y \hookrightarrow H^{-\frac{1}{2}}(\Sigma)$ . By the definition of  $\Gamma_\nu$ , this can be written:

$$(2.22) \quad i_Y^* \nu u = (L + i_Y^* P_{\gamma,\nu}) \gamma_0 u.$$

In the case where  $X = Y = H^{-\frac{1}{2}}(\Sigma)$ , this is simply a Neumann-type condition

$$(2.23) \quad \nu u = C \gamma_0 u, \text{ where } C = L + P_{\gamma,\nu}.$$

In the present paper we are more interested in a genuine subspace case, where  $X = H_0^{-\frac{1}{2}}(\Sigma_+)$ ; we return to that below.

## 2.4 Neumann reference operator.

For the abstract theory using  $A_\nu$  as the reference operator, we get slightly different but analogous formulas:

Let  $\tilde{A}$  correspond to  $T_1: V_1 \rightarrow W_1$  as in (2.12). We now set  $X_1 = \nu(V_1)$ ,  $Y_1 = \nu(W_1)$ , closed subspaces of  $H^{-\frac{3}{2}}(\Sigma)$ , and denote the connecting homeomorphisms

$$(2.24) \quad \nu_{V_1}: V_1 \xrightarrow{\sim} X_1, \quad \nu_{W_1}: W_1 \xrightarrow{\sim} Y_1.$$

Now  $T_1: V_1 \rightarrow W_1$  is carried over to the map  $L_1: X_1 \rightarrow Y_1^*$  defined by

$$(2.25) \quad L_1 = (\nu_{W_1}^*)^{-1} T_1 \nu_{V_1}^{-1}.$$

In the invertible case, the abstract resolvent formula (2.13) carries over to the formula:

$$(2.26) \quad \tilde{A}^{-1} = A_\nu^{-1} + K_{\nu, X_1} L_1^{-1} K_{\nu, Y_1}^*$$

where  $K_{\nu, X_1} = i_{V_1} \nu_{V_1}^{-1}: X_1 \rightarrow V_1 \subset H$ ,  $(K_{\nu, Y_1})^* = (\nu_{W_1}^*)^{-1} \text{pr}_{W_1}: H \rightarrow Y_1$ ; another Kreĭn resolvent formula. In particular, if  $V_1 = W_1 = Z$ , then  $X_1 = Y_1 = H^{-\frac{3}{2}}(\Sigma)$ , and (2.26) takes the form

$$(2.27) \quad \tilde{A}^{-1} = A_\nu^{-1} + K_\nu L_1^{-1} K_\nu^*,$$

where  $L_1$  goes from  $D(L_1) \subset H^{-\frac{3}{2}}(\Sigma)$  to  $H^{\frac{3}{2}}(\Sigma)$ .

The interpretation of  $\tilde{A}$  as defined by a boundary condition is here based on the second line of (2.20) and goes as follows:  $D(\tilde{A})$  consists of the functions  $u \in D(A_{\max})$  that satisfy the boundary condition

$$(2.28) \quad \nu u \in D(L_1), \quad -(\Gamma_\gamma u, \varphi)_{\frac{3}{2}, -\frac{3}{2}} = (L_1 \nu u, \varphi)_{Y_1^*, Y_1} \text{ for all } \varphi \in Y_1.$$

Here the second condition is rewritten as  $i_{Y_1}^* \Gamma_\gamma u = -L_1 \nu u$ , or

$$(2.29) \quad i_{Y_1}^* \gamma_0 u = (-L_1 + i_{Y_1}^* P_{\nu, \gamma}) \nu u.$$

In the case where  $X_1 = Y_1 = H^{-\frac{3}{2}}(\Sigma)$ , this is a ‘‘Dirichlet-type’’ condition

$$(2.30) \quad \gamma_0 u = C_1 \nu u, \text{ where } C_1 = -L_1 + P_{\nu, \gamma}.$$

We shall see later that the mixed problem can be written in this form (after a replacement of  $\nu$  by  $\nu + K\gamma_0$ , if necessary).

In the above analysis we assumed  $A_\gamma$  resp.  $A_\nu$  positive, so that  $0 \in \varrho(A_\gamma)$  resp.  $0 \in \varrho(A_\nu)$ . Clearly, by addition of real constants to  $A$  this covers the realizations of  $A - \lambda$  for  $-\lambda$  large positive. The formulation was just chosen for simplicity of notation; the theory of [G68] in fact works for any  $\lambda \in \varrho(A_\gamma)$  resp.  $\lambda \in \varrho(A_\nu)$ . For general  $\lambda$  one uses the nullspaces  $Z_\lambda = \ker(A_{\max} - \lambda)$  and  $Z_{\bar{\lambda}} = \ker(A_{\max} - \bar{\lambda})$ . For the various spaces, operators and auxiliary Poisson, pseudodifferential and trace operators, the  $\lambda$ -dependence is indicated by

$$(2.31) \quad V_\lambda, W_{\bar{\lambda}}, L^\lambda, K_\gamma^\lambda, K_{\bar{\gamma}}^{\bar{\lambda}}, P_{\gamma, \nu}^\lambda, P_{\nu, \gamma}^\lambda, \Gamma_\nu^\lambda, \text{ etc.}$$

The  $\lambda$ -dependent formulas are explained in detail in [BGW09] (based on methods from [G74]), see also [AGW11] for notation. There is an important point here, namely that  $X = \gamma_0 V_\lambda$  and  $Y = \gamma_0 W_{\bar{\lambda}}$  are *independent of  $\lambda$* . Moreover  $D(L^\lambda) = D(L^0)$ , and  $L^\lambda - L^0$  acts as the *bounded* operator  $i_Y^* (P_{\gamma, \nu}^0 - P_{\gamma, \nu}^\lambda)$ . Related statements hold for  $L_1^\lambda: X_1 \rightarrow Y_1$ . The Kreĭn resolvent formulas have the form:

$$(2.32) \quad \begin{aligned} (\tilde{A} - \lambda)^{-1} &= (A_\gamma - \lambda)^{-1} + K_{\gamma, X}^\lambda (L^\lambda)^{-1} (K_{\gamma, Y}^{\bar{\lambda}})^* \text{ when } \lambda \in \varrho(A_\gamma) \cap \varrho(\tilde{A}), \\ (\tilde{A} - \lambda)^{-1} &= (A_\nu - \lambda)^{-1} + K_{\nu, X_1}^\lambda (L_1^\lambda)^{-1} (K_{\nu, Y_1}^{\bar{\lambda}})^* \text{ when } \lambda \in \varrho(A_\nu) \cap \varrho(\tilde{A}). \end{aligned}$$

Other Kreĭn resolvent formulas have been established e.g. in Malamud and Mogilevskii [MM02], [M10], Pankrashkin [P06], Behrndt and Langer [BL07], Alpay and Behrndt [AB09], Gesztesy and Mitrea [GM08, GM09, GM11], Brown, Marletta, Naboko and Wood [BMNW08], Posilicano and Raimondi [PR09].



**Remark 2.2.** The theory recalled above has, in the study of “pure” boundary conditions (of Neumann-type  $\nu u = C\gamma_0 u$  or of Dirichlet-type  $\gamma_0 u = C_1 \nu u$ ), much in common with the representations of boundary value problems based on boundary triples theory. It is when *subspaces*  $V, W$  of  $Z$  occur that our theory differs markedly from the others, which obtain a generalization by allowing *relations* instead of operators.

### 3. BIRMAN’S METHOD REVISITED

The correspondence (2.8) with  $A_\gamma$  as reference operator is used here. We have that  $D(A_\gamma) = H_0^1(\Omega) \cap H^2(\Omega)$  and  $D(a_\gamma) = H_0^1(\Omega)$ . For  $A_\chi$ , the decomposition in (2.11) gives  $D(a_\chi) = H^1(\Omega) = H_0^1(\Omega) \dot{+} Z^1$ , where  $Z^1 = Z \cap H^1(\Omega)$ . The corresponding operator  $T_\chi$  is defined from the sesquilinear form  $t_\chi$  obtained by restricting  $a_\chi$  to  $Z^1$  in  $Z$ ;  $T_\chi$  is a selfadjoint unbounded positive operator in  $Z$  with domain dense in  $Z^1$ . For the mixed problem,  $D(a_{\chi, \Sigma_+}) = H_0^1(\Omega) \dot{+} Z_{\Sigma_+}^1$ , where  $Z_{\Sigma_+}^1 = Z \cap H_{\Sigma_+}^1(\Omega)$  (cf. (2.3)); the corresponding operator  $T_{\chi, \Sigma_+}$  is a selfadjoint operator in  $Z_{\Sigma_+} = \overline{Z_{\Sigma_+}^1}$  ( $L_2(\Omega)$ -closure) with domain dense in  $Z_{\Sigma_+}^1$ .

There are bounded, in fact compact, inverses  $T_\chi^{-1}$  on  $Z$ , resp.  $T_{\chi, \Sigma_+}^{-1}$  on  $Z_{\Sigma_+}$ .

When a general  $T$  is derived from the form  $t = \tilde{a}|_{D(t)}$  and  $T^{-1}$  is compact nonnegative, then the eigenvalues are determined by the minimum-maximum principle from Rayleigh quotients:

$$(3.1) \quad \mu_j(T^{-1}) = \min_{U \subset D(t), \dim U = j-1} \max_{z \perp U, z \in D(t) \setminus \{0\}} \frac{\|z\|_0^2}{\tilde{a}(z, z)}.$$

This principle was used in Birman [B62] to reduce the proof of upper estimates of the  $\mu_j(T^{-1})$  for each of the boundary conditions (1.5) to simpler cases where it could be found by computation.

We shall here show how the principle leads to a lim sup estimate for the mixed problem. Consider  $a_{\chi, \Sigma_+}$  and the Robin case  $a_{\chi_K}$  (where  $b$  is replaced by  $-K$ , cf. Section 2.1). Let the corresponding operators and forms defined on subspaces of  $Z$  be denoted  $T_{\chi, \Sigma_+}$  and  $T_{\chi_K}$ , resp.  $t_{\chi, \Sigma_+}$  and  $t_{\chi_K}$ . Here  $D(t_{\chi_K}) = Z^1$ , and  $D(t_{\chi, \Sigma_+}) = Z_{\Sigma_+}^1 \subset Z^1$ . Then

$$(3.2) \quad \begin{aligned} \mu_j(T_{\chi, \Sigma_+}^{-1}) &= \min_{U \subset Z_{\Sigma_+}^1, \dim U = j-1} \max_{z \perp U, z \in Z_{\Sigma_+}^1 \setminus \{0\}} \frac{\|z\|_0^2}{a(z, z) + (b\gamma_0 z, \gamma_0 z)_{\Sigma_+}} \\ &\leq \min_{U \subset Z^1, \dim U = j-1} \max_{z \perp U, z \in Z^1 \setminus \{0\}} \frac{\|z\|_0^2}{a(z, z) + (b\gamma_0 z, \gamma_0 z)_{\Sigma_+}} \\ &\leq \min_{U \subset Z^1, \dim U = j-1} \max_{z \perp U, z \in Z^1 \setminus \{0\}} \frac{\|z\|_0^2}{a(z, z) - K\|\gamma_0 z\|_{L_2(\Sigma)}^2} = \mu_j(T_{\chi_K}^{-1}). \end{aligned}$$

Birman showed in [B62] that  $\mu_j(T_{\chi_K}^{-1})$ , and hence also  $\mu_j(T_{\chi, \Sigma_+}^{-1})$ , is  $O(j^{-2/(n-1)})$  for  $j \rightarrow \infty$ . It is noteworthy that this included the mixed problem.

In the finer asymptotic estimate (1.7)–(1.8),  $p^0(x', \xi')$  denotes the principal symbol of  $P_{\gamma, \nu}$  and  $\tilde{k}^0(x', \xi', \xi_n)$  is the principal symbol-kernel of  $K_\gamma$ ; the derivation of the formula is explained in [G11a], Th. 2.4. Applying (1.7)–(1.8) to  $T_{\chi_K}$  we can now get a lim sup estimate using (3.2):

**Proposition 3.1.** *The nonzero eigenvalues of  $A_{\chi, \Sigma_+}^{-1} - A_\gamma^{-1}$  satisfy, with  $C_0$  from (1.8),*

$$(3.3) \quad \limsup_{j \rightarrow \infty} \mu_j(A_{\chi, \Sigma_+}^{-1} - A_\gamma^{-1}) j^{2/(n-1)} \leq C_0^{2/(n-1)}.$$

*Proof.* From (1.7) with  $b = -K$  follows in view of (3.2):

$$(3.4) \quad \begin{aligned} \limsup_{j \rightarrow \infty} \mu_j(T_{\chi, \Sigma_+}^{-1}) j^{2/(n-1)} &\leq \limsup_{j \rightarrow \infty} \mu_j(T_{\chi_K}^{-1}) j^{2/(n-1)} \\ &= \lim_{j \rightarrow \infty} \mu_j(A_{\chi_K}^{-1} - A_\gamma^{-1}) j^{2/(n-1)} = C_0^{2/(n-1)}; \end{aligned}$$

we have here applied formula (2.10) with  $\tilde{A} = A_{\chi_K}$ . Similarly,  $A_{\chi, \Sigma_+}^{-1} - A_\gamma^{-1}$  and  $T_{\chi, \Sigma_+}^{-1}$  have the same nonzero eigenvalues, so the result follows.  $\square$

We also get a spectral estimate for the eigenvalues of  $A_{\chi, \Sigma_+}^{-1}$  itself:

**Corollary 3.2.** *The eigenvalues of  $A_{\chi, \Sigma_+}$  satisfy:*

$$(3.5) \quad \mu_j(A_{\chi, \Sigma_+}^{-1}) - C_A^{2/n} j^{-2/n} \text{ is } O(j^{-(1+1/(n+1))2/n}) \text{ for } j \rightarrow \infty,$$

where

$$(3.6) \quad C_A = (2\pi)^{-n} \int_{x \in \Omega, a^0(x, \xi) < 1} dx d\xi.$$

*Proof.* It is known (cf. e.g. [H85], Sect. 29.3) that the spectrum of  $A_\gamma$  satisfies the asymptotic estimate

$$(3.7) \quad \mu_j(A_\gamma^{-1}) - C_A^{2/n} j^{-2/n} \text{ is } O(j^{-3/n}) \text{ for } j \rightarrow \infty,$$

with  $C_A$  defined by (3.6) (the spectral estimate is formulated for the counting function in [H85], but carries over to the above form, cf. e.g. [G96], Lemma A.5). We shall apply a perturbation result to this estimate, using (3.3) and (2.10) with  $\tilde{A} = A_{\chi, \Sigma_+}$ .

Recall from [G84], Prop. 6.1 (or [G96], Lemma A.6), that when  $B$  and  $B'$  are compact operators satisfying for  $j \rightarrow \infty$ , with  $p > q > 0$ ,  $p > r > 0$ ,  $c_0 \geq 0$ ,

$$(3.8) \quad s_j(B) - c_0^{1/p} j^{-1/p} \text{ is } O(j^{-1/q}), \quad s_j(B') \text{ is } O(j^{-1/r}),$$

then  $B + B'$  satisfies

$$(3.9) \quad s_j(B + B') - c_0^{1/p} j^{-1/p} \text{ is } O(j^{-1/q'}), \text{ with } q' = \max \left\{ q, p \frac{r+1}{p+1} \right\}.$$

We apply the result here with  $B = A_\gamma^{-1}$  and  $B' = A_{\chi, \Sigma_+}^{-1} - A_\gamma^{-1}$ , so that  $p = n/2$ ,  $q = n/3$ ,  $r = (n-1)/2$ . This gives

$$q' = \max \left\{ \frac{n}{3}, \frac{n}{2} \cdot \frac{\frac{n-1}{2} + 1}{\frac{n}{2} + 1} \right\} = \frac{n}{2} \cdot \frac{n+1}{n+2},$$

here  $1/q' = 2/n \cdot (n+2)/(n+1) = (1 + 1/(n+1))2/n$ .  $\square$

Note that these results hold when  $b \in L_\infty(\Sigma)$  and  $\Sigma_+$  is any closed subset of  $\Sigma$ .

**Remark 3.3.** Concerning nonsmooth choices of  $\Omega$ , let us mention that the basic hypothesis of Birman in [B62] is that  $\Sigma$  is piecewise  $C^2$ . This allows edges or creases, cf. Section 4.3 below. Moreover, in the recent translation to English of that historical paper, the translator M. Solomyak states in a supplementing comment to Section 2.1 on page 50 that the result is valid for Lipschitz domains (at least when  $n \geq 3$ ; the reservation for  $n = 2$  seems to be concerned with cutoffs in exterior domains).

#### 4. KREĬN RESOLVENT FORMULAS FOR THE MIXED PROBLEM

##### 4.1 A formula relative to the Dirichlet problem.

We assume in Sections 4.1, 4.2 and 5 that  $\Sigma_+$  is smooth. First we show a Kreĭn resolvent formula for  $A_{\chi, \Sigma_+}$  linked with  $A_\gamma$ . For simplicity of notation, we do the main calculations in the case  $\lambda = 0$  (where the indexation by  $\lambda$  is left out); then at the end we account for the consequences in situations with other values of  $\lambda$ .

Recall from Section 2.3 that in the analysis with  $A_\gamma$  as the reference operator,  $A_{\chi, \Sigma_+}$  corresponds to  $L: X \rightarrow X^*$ , where  $D(L) = \gamma_0 D(A_{\chi, \Sigma_+})$  and  $X$  is its closure in  $H^{-\frac{1}{2}}(\Sigma)$ . It is seen from (2.6) that  $D(L)$  is a subset of  $H_0^{\frac{1}{2}}(\Sigma_+)$ , and it contains  $C_0^\infty(\Sigma_+^\circ)$  in view of the surjectiveness of  $\{\gamma_0, \nu\}$  from  $H^2(\Omega)$  to  $H^{\frac{3}{2}}(\Sigma) \times H^{\frac{1}{2}}(\Sigma)$ . Then in fact its closure  $X$  in  $H^{-\frac{1}{2}}(\Sigma)$  satisfies

$$(4.1) \quad X = H_0^{-\frac{1}{2}}(\Sigma_+), \text{ and hence } X^* = H^{\frac{1}{2}}(\Sigma_+^\circ).$$

We note that the injection  $i_X: X \hookrightarrow H^{-\frac{1}{2}}(\Sigma)$  and its adjoint satisfy:

$$i_X = e_{\Sigma_+^\circ}: H_0^{-\frac{1}{2}}(\Sigma_+) \hookrightarrow H^{-\frac{1}{2}}(\Sigma), \quad (i_X)^* = r_{\Sigma_+^\circ}: H^{\frac{1}{2}}(\Sigma) \rightarrow H^{\frac{1}{2}}(\Sigma_+^\circ),$$

where  $e_{\Sigma_+^\circ}$  is a well-defined extension of the operator that extends functions on  $\Sigma_+^\circ$  by zero on  $\Sigma_-$ , and  $r_{\Sigma_+^\circ}$  denotes restriction to  $\Sigma_+^\circ$ . We denote  $e_{\Sigma_+^\circ} = e^+$  and  $r_{\Sigma_+^\circ} = r^+$  for short. Since  $A_{\chi, \Sigma_+}$  is bijective, so is  $L$ , from  $D(L)$  to  $H^{\frac{1}{2}}(\Sigma_+^\circ)$ .

When  $u \in D(A_{\chi, \Sigma_+})$ , we see from (2.6), (1.3) that  $\nu u$  equals  $b\gamma_0 u$  on  $\Sigma_+^\circ$  in the distribution sense, hence since  $\Gamma_\nu = \nu - P_{\gamma, \nu}\gamma_0$ ,

$$(4.2) \quad \langle \Gamma_\nu u, \bar{\zeta} \rangle = \langle (b - P_{\gamma, \nu})\gamma_0 u, \bar{\zeta} \rangle \text{ for } \zeta \in C_0^\infty(\Sigma_+^\circ).$$

Since  $\gamma_0 u \in H^{\frac{1}{2}}(\Sigma)$ , which is mapped to  $H^{-\frac{1}{2}}(\Sigma)$  by  $P_{\gamma, \nu}$ , and multiplication by  $b$  preserves  $L_2(\Sigma)$ , we have that  $(b - P_{\gamma, \nu})\gamma_0 u \in H^{-\frac{1}{2}}(\Sigma)$ .

The operator  $L$  satisfies, by (2.21),

$$(L\gamma_0 u, \varphi)_{X^*, X} = (\Gamma_\nu u, \varphi)_{\frac{1}{2}, -\frac{1}{2}} \text{ for all } \varphi \in X;$$

in particular, when (4.1) and (4.2) are taken into account,

$$(L\gamma_0 u, \zeta)_{H^{\frac{1}{2}}(\Sigma_+^\circ), H_0^{-\frac{1}{2}}(\Sigma_+)} = \langle (b - P_{\gamma, \nu})\gamma_0 u, \bar{\zeta} \rangle, \text{ for } \zeta \in C_0^\infty(\Sigma_+^\circ),$$

so

$$L\gamma_0 u = r^+(b - P_{\gamma, \nu})\gamma_0 u, \text{ for } u \in D(A_{\chi, \Sigma_+}).$$

Thus  $L$  acts as

$$(4.3) \quad L\varphi = r^+(b - P_{\gamma, \nu})e^+\varphi, \text{ for } \varphi \in D(L).$$

This shows the form of  $L$ . We need deeper theories to say more about the domain. Here we shall use the study of mixed problems in Shamir [S68]; in Section 5 we also use Eskin [E81]. Some smoothness is needed for this; for convenience we take  $b \in C^\infty(\Sigma)$ .

**Proposition 4.1.** *When  $\Sigma_+$  is smooth, the operator  $L$  acts as in (4.3). When also  $b$  is smooth, it satisfies*

$$(4.4) \quad D(L) \subset H_0^{1-\varepsilon}(\Sigma_+), \text{ any } \varepsilon > 0,$$

and  $L^{-1}$  maps  $H^{\frac{1}{2}}(\Sigma_+^\circ)$  into  $H_0^{1-\varepsilon}(\Sigma_+)$ .

*Proof.* We see from [S68] that  $D(A_{\chi, \Sigma_+}) \subset H^{\frac{3}{2}-\varepsilon}(\Omega)$ , as follows: First Shamir shows this in Th. 3.1 of [S68] for the constant-coefficient case of  $-\Delta + \alpha^2$  on a half-space with mixed Dirichlet and Neumann boundary conditions. Subsequently the statement is extended to variable coefficients and bounded domains in the proof of Lemma 5.1 of [S68] (when we recall that the domain is a priori contained in  $H^1(\Omega)$ ). Since  $\gamma_0 H^{\frac{3}{2}-\varepsilon}(\Omega) = H^{1-\varepsilon}(\Sigma)$ , it follows by the definition of  $L$  that  $D(L) \subset H_0^{1-\varepsilon}(\Sigma_+)$ . Since  $L$  is surjective onto  $H^{\frac{1}{2}}(\Sigma_+^\circ)$ , the last statement follows.  $\square$

There is a simple example mentioned in [S68] of a harmonic function  $u(x_1, x_2) = \text{Im}(x_2 + ix_1)^{\frac{1}{2}}$  on  $\mathbb{R}_+ \times \mathbb{R}$  satisfying the mixed condition on  $\{x_1 = 0\}$ , namely  $\gamma_0 u = 0$  for  $x_2 > 0$ ,  $\gamma_1 u = 0$  for  $x_2 < 0$ . It is not in  $H^{\frac{3}{2}}$  in a neighborhood of 0. This shows that  $D(A_{\chi, \Sigma_+})$  is not in general contained in  $H^{\frac{3}{2}}(\Omega)$ , so the regularity cannot be improved.

Now consider the Kreĭn resolvent formula (2.16) for this choice of  $L$  and  $X$ ; by the selfadjointness,  $Y = X$ . Recall that  $K_{\gamma,X} = i_V \gamma_V^{-1}: X \rightarrow L_2(\Omega)$ , where  $V$  is the subspace of  $Z = \ker(A_{\max})$  that is mapped to  $X$  by  $\gamma_0$ . Since  $\gamma_V^{-1}$  acts like  $K_\gamma$  from the space  $X = H_0^{-\frac{1}{2}}(\Sigma_+)$  to  $V$ , we can also write

$$K_{\gamma,X} = i_V K_\gamma e^+: H_0^{-\frac{1}{2}}(\Sigma_+) \rightarrow L_2(\Omega), \text{ and then } K_{\gamma,X}^* = r^+ K_\gamma^* \text{pr}_V: L_2(\Omega) \rightarrow H^{\frac{1}{2}}(\Sigma_+^\circ),$$

whereby the formula takes the form

$$(4.5) \quad A_{\chi,\Sigma_+}^{-1} - A_\gamma^{-1} = i_V K_\gamma e^+ L^{-1} r^+ K_\gamma^* \text{pr}_V = i_V K_\gamma e^+ (r^+ (b - P_{\gamma,\nu}) e^+)^{-1} r^+ K_\gamma^* \text{pr}_V.$$

The  $\lambda$ -dependent version is found by replacing  $A$  by  $A - \lambda$  in the various defining formulas, as explained at the end of Section 2. Since  $\chi = \nu - b\gamma_0$ , we have that

$$(4.6) \quad P_{\gamma,\chi}^\lambda = \chi K_\gamma^\lambda = P_{\gamma,\nu}^\lambda - b,$$

a notation we shall now use. Moreover, using the standard abbreviation for a truncated operator  $r^+ Q e^+ = Q_+$ , we can write  $r^+ P_{\gamma,\chi}^\lambda e^+ = P_{\gamma,\chi,+}^\lambda$ . Then the result in the  $\lambda$ -dependent formulation is:

**Theorem 4.2.** *Let  $\Sigma_+$  and  $b$  be smooth. Then*

$$(4.7) \quad \begin{aligned} L^\lambda \varphi &= -P_{\gamma,\chi,+}^\lambda \varphi \text{ for } \varphi \in D(L), \lambda \in \varrho(A_\gamma), \\ (A_{\chi,\Sigma_+} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1} &= -K_{\gamma,X}^\lambda (P_{\gamma,\chi,+}^\lambda)^{-1} (K_{\gamma,X}^\lambda)^* \text{ for } \lambda \in \varrho(A_{\chi,\Sigma_+}) \cap \varrho(A_\gamma). \end{aligned}$$

where  $V_\lambda = K_\gamma^\lambda(X)$ ,  $K_{\gamma,X}^\lambda = i_{V_\lambda} \gamma_{V_\lambda}^{-1} = i_{V_\lambda} K_\gamma^\lambda e^+$ ,  $(K_{\gamma,X}^\lambda)^* = (\gamma_{V_\lambda}^{-1})^* \text{pr}_{V_\lambda} = r^+ (K_\gamma^\lambda)^* \text{pr}_{V_\lambda}$ .

The inverse of  $P_{\gamma,\chi}^\lambda$  is  $P_{\chi,\gamma}^\lambda$ , when it exists. It is important to observe that  $(P_{\gamma,\chi,+}^\lambda)^{-1}$  is *not* the same as  $P_{\chi,\gamma,+}^\lambda$ ; this is part of the difficulty treated in Section 5.

#### 4.2 Other Kreĭn resolvent formulas.

Next, if we work instead with a Neumann realization as the reference operator, we can show a different formula containing full Poisson operators.

Consider again the boundary condition

$$(4.8) \quad \gamma_0 u = 0 \text{ on } \Sigma_-, \quad \nu u = b \gamma_0 u \text{ on } \Sigma_+.$$

If  $b$  has a bounded inverse  $f$ , we can set  $f_+ = 1_{\Sigma_+} f$  and write condition (4.8) as one equation, a Dirichlet-type condition

$$(4.9) \quad \gamma_0 u = f_+ \nu u.$$

Here  $\gamma_0 u$  is a function of  $\nu u$ , so that the operator  $A_\nu$  can be used in a simple way as the reference operator.

Actually, it only takes a small modification to obtain invertibility of the coefficient in general: If  $b$  does not have a bounded inverse, we can replace  $\nu u$  by

$$(4.10) \quad \nu' u = \nu u + K \gamma_0 u,$$

where  $K$  is chosen  $> \text{ess sup } |b(x)|$  (as in Section 2.1); then the condition (4.8) takes the form

$$(4.11) \quad \gamma_0 u = 0 \text{ on } \Sigma_-, \quad \nu' u = b' \gamma_0 u \text{ on } \Sigma_+,$$

where  $b' = b + K$  does have a bounded inverse. Note that  $\chi' = \nu' - b' \gamma_0 = \chi$  by cancellation. In Green's formula (2.14) we get  $\nu$  replaced by  $\nu'$  by adding the term  $(K \gamma_0 u, \gamma_0 u) - (\gamma_0 u, K \gamma_0 u)$  (equal

to 0) to the right-hand side, and the sesquilinear form is adapted to these formulas by addition of the first-order terms  $\sum_{j=1}^n [(Kn_j \partial_j u, v)_\Omega + (u, K \partial_j (n_j v))_\Omega]$ , giving the form

$$a'(u, v) = a(u, v) + \sum_{j=1}^n [(Kn_j \partial_j u, v) + (u, K \partial_j (n_j v))] \text{ on } H^1(\Omega).$$

Here the  $n_j$  are extended smoothly to the interior of  $\Omega$ , vanishing outside a small neighborhood of  $\Sigma$ . The operators defined from  $a'$  on various spaces between  $H^1(\Omega)$  and  $H_0^1(\Omega)$  still act like  $A$ , since  $(u, K \partial_j (n_j \varphi)) = -(Kn_j \partial_j u, \varphi)$  for  $\varphi \in C_0^\infty(\Omega)$ . The “halfways Green’s formula” is here

$$(4.12) \quad (Au, v) - a'(u, v) = (\nu' u, \gamma_0 v)_{L_2(\Sigma)},$$

since  $\sum_{j=1}^n [(Kn_j \partial_j u, v)_\Omega + (u, K \partial_j (n_j v))_\Omega] = -(K \sum n_j^2 \gamma_0 u, \gamma_0 v)_\Sigma = -(K \gamma_0 u, \gamma_0 v)_\Sigma$ . The forms in the scheme (2.2) are now replaced by

$$(4.13) \quad \begin{aligned} a'_\gamma(u, v) &= a'(u, v) \text{ on } H_0^1(\Omega), \text{ leading to } A_\gamma, \\ a'_{\nu'}(u, v) &= a'(u, v) \text{ on } D(a_{\nu'}) = H^1(\Omega), \text{ leading to } A_{\nu'}, \\ a'_{\chi'}(u, v) &= a'(u, v) + (b' \gamma_0 u, \gamma_0 v)_{L_2(\Sigma)} \text{ on } D(a_{\chi'}) = H^1(\Omega), \text{ leading to } A_\chi, \\ a'_{\chi', \Sigma_+}(u, v) &= a'(u, v) + (b' \gamma_0 u, \gamma_0 v)_{L_2(\Sigma_+)} \text{ on } D(a'_{\chi', \Sigma_+}) = H_{\Sigma_+}^1(\Omega) \\ &= \{u \in H^1(\Omega) \mid \text{supp } u \subset \Sigma_+\}, \text{ leading to } A_{\chi, \Sigma_+}; \end{aligned}$$

here  $A_{\nu'}$  is the realization of  $A$  under the boundary condition  $\nu' u = 0$ , whereas the choices with  $b'$  still give the boundary condition  $\nu u = b \gamma_0 u$  on  $\Sigma$  resp.  $\Sigma_+$ , since  $b' = b + K$ ,  $\nu' = \nu + K \gamma_0$ . With  $K_{\nu'}$ ,  $P_{\gamma, \nu'}$  and  $P_{\nu', \gamma}$  defined as in Section 4.1 with  $\nu$  replaced by  $\nu'$ , and

$$\Gamma'_\gamma = \gamma_0 - P_{\nu', \gamma} \nu',$$

we have the generalized Green’s formula valid for  $u, v \in D(A_{\max})$ :

$$(4.14) \quad (Au, v)_{L_2(\Omega)} - (u, Av)_{L_2(\Omega)} = (\nu' u, \Gamma'_\gamma v)_{-\frac{3}{2}, \frac{3}{2}} - (\Gamma'_\gamma u, \nu' v)_{\frac{3}{2}, -\frac{3}{2}}.$$

In the following, we assume that the forms in (4.13) all have positive lower bound. We set  $f = (b')^{-1}$  so that the mixed boundary condition (4.11) can be written

$$(4.15) \quad \gamma_0 u = f_+ \nu' u,$$

where  $f_+ = 1_{\Sigma_+} f$ , as accounted for above.

We now describe  $A_{\chi, \Sigma_+}$  in terms of the correspondence (2.12) and its interpretation in Section 2.4, with  $\nu$  replaced by  $\nu'$ .

Here  $X_1$  is the full space  $H^{-\frac{3}{2}}(\Sigma)$ , which is seen as follows: When  $\psi \in C_0^\infty(\Sigma_+^\circ) \cup C_0^\infty(\Sigma_-^\circ)$ , then  $f_+ \psi \in C_0^\infty(\Sigma_+^\circ)$ , and there exists  $u \in C^\infty(\overline{\Omega})$  such that  $\nu' u = \psi$ ,  $\gamma_0 u = f_+ \psi$ ; this  $u$  satisfies (4.9). So  $C_0^\infty(\Sigma_+^\circ) \cup C_0^\infty(\Sigma_-^\circ) \subset D(L_1)$ . It is known that  $C_0^\infty(\Sigma_+^\circ) \cup C_0^\infty(\Sigma_-^\circ)$  is dense in  $H^s(\Sigma)$  for  $s < \frac{1}{2}$ . In particular  $C_0^\infty(\Sigma_+^\circ) \cup C_0^\infty(\Sigma_-^\circ)$  is dense in  $H^{-\frac{3}{2}}(\Sigma)$ , so we conclude that  $X_1 = H^{-\frac{3}{2}}(\Sigma)$ . Since  $A_{\chi, \Sigma_+}$  is selfadjoint, also  $Y_1 = H^{-\frac{3}{2}}(\Sigma)$ .

Thus the realization  $A_{\chi, \Sigma_+}$  with domain (2.6) corresponds to an operator  $L_1: H^{-\frac{3}{2}}(\Sigma) \rightarrow H^{\frac{3}{2}}(\Sigma)$  with domain  $D(L_1) = \nu' D(A_{\chi, \Sigma_+})$ ; the latter lies in  $H^{-\frac{1}{2}}(\Sigma)$  since  $D(A_{\chi, \Sigma_+}) \subset H^1(\Omega)$ . It follows by comparison of (2.30) with (4.15) that  $L_1$  acts as

$$(4.16) \quad L_1 = -f_+ + P_{\nu', \gamma}.$$

Since  $A_{\chi, \Sigma_+}$  is bijective, so is  $L_1$ .

Then the Kreĭn resolvent formula reads

$$(4.17) \quad A_{\chi, \Sigma_+}^{-1} - A_{\nu'}^{-1} = K_{\nu'} L_1^{-1} K_{\nu'}^* = K_{\nu'} (P_{\nu', \gamma} - f_+)^{-1} K_{\nu'}^*, \text{ for } \lambda \in \varrho(A_{\chi, \Sigma_+}) \cap \varrho(A_{\nu'}).$$

It may look a little more useful than (4.5), since the operators surrounding  $L_1^{-1}$  are a full Poisson operator and trace operator in the pseudodifferential boundary operator calculus, but it poses again the question of a detailed understanding of the term in the middle, defined on  $H^{\frac{3}{2}}(\Sigma)$ . This may not be any easier than our treatment in Section 4.1, since the principal part of  $L_1$  is the 0-order multiplication by  $-f_+$  which vanishes on  $\Sigma_-$ , and  $P_{\nu', \gamma}$  is of order  $-1$ .

We can replace  $A$  by  $A - \lambda$  in the various defining formulas and obtain:

**Theorem 4.3.** *Let  $\Sigma_+$  and  $b$  be smooth. Define  $\nu'$  by (4.10)ff. and  $f = (b + K)^{-1}$ . Then*

$$(4.18) \quad (A_{\chi, \Sigma_+} - \lambda)^{-1} - (A_{\nu'} - \lambda)^{-1} = K_{\nu'}^\lambda (P_{\nu', \gamma}^\lambda - f_+)^{-1} (K_{\nu'}^\lambda)^*, \text{ for } \lambda \in \varrho(A_{\nu'}) \cap \varrho(A_{\chi, \Sigma_+}).$$

Formula (4.18) can even be turned into a resolvent difference formula where the surrounding Poisson operator and trace operator come from the Dirichlet problem, by use of the fact that  $P_{\gamma, \nu'}^\lambda$  and  $P_{\nu', \gamma}^\lambda$  are inverses of one another, and

$$(4.19) \quad K_{\nu'}^\lambda = K_\gamma^\lambda P_{\nu', \gamma}^\lambda, \quad (P_{\nu', \gamma}^\lambda)^* = P_{\nu', \gamma}^\lambda,$$

then. Namely, insertion in (4.18) gives:

$$(4.20) \quad (A_{\chi, \Sigma_+} - \lambda)^{-1} - (A_{\nu'} - \lambda)^{-1} = K_\gamma^\lambda P_{\nu', \gamma}^\lambda (P_{\nu', \gamma}^\lambda - f_+)^{-1} P_{\nu', \gamma}^\lambda (K_\gamma^\lambda)^*.$$

This can be added to the well-known formula

$$(A_{\nu'} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1} = K_\gamma^\lambda (-P_{\gamma, \nu'}^\lambda)^{-1} (K_\gamma^\lambda)^* = -K_\gamma^\lambda P_{\nu', \gamma}^\lambda (K_\gamma^\lambda)^*$$

((2.32) with  $\tilde{A} = A_{\nu'}$ , hence  $L^\lambda = -P_{\gamma, \nu'}^\lambda$ ), to give a formula for the resolvent difference with the Dirichlet realization, having another structure than (4.7):

$$(4.21) \quad (A_{\chi, \Sigma_+} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1} = K_\gamma^\lambda P_{\nu', \gamma}^\lambda [(P_{\nu', \gamma}^\lambda - f_+)^{-1} P_{\nu', \gamma}^\lambda - 1] (K_\gamma^\lambda)^* \\ = K_\gamma^\lambda P_{\nu', \gamma}^\lambda (P_{\nu', \gamma}^\lambda - f_+)^{-1} f_+ (K_\gamma^\lambda)^*, \text{ for } \lambda \in \varrho(A_{\chi, \Sigma_+}) \cap \varrho(A_\gamma) \cap \varrho(A_{\nu'}).$$

The last formula in (4.21) has a similar flavor as the formula found by Pankrashkin in [P06], Sect. 4.3.

If  $b$  itself is invertible, the formulas will be valid with  $f = b^{-1}$ ,  $\nu'$  replaced by  $\nu$ . We have shown:

**Corollary 4.4.** *Under the hypotheses of Theorem 4.3, we have the formulas in (4.21) for the difference with the Dirichlet resolvent, when  $\lambda \in \varrho(A_{\chi, \Sigma_+}) \cap \varrho(A_\gamma) \cap \varrho(A_{\nu'})$ .*

*If  $b$  itself is invertible, there are the formulas with  $f = b^{-1}$ :*

$$(4.22) \quad (A_{\chi, \Sigma_+} - \lambda)^{-1} - (A_\nu - \lambda)^{-1} = K_\nu^\lambda (P_{\nu, \gamma}^\lambda - f_+)^{-1} (K_\nu^\lambda)^*,$$

$$(4.23) \quad (A_{\chi, \Sigma_+} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1} = K_\gamma^\lambda P_{\nu, \gamma}^\lambda (P_{\nu, \gamma}^\lambda - f_+)^{-1} f_+ (K_\gamma^\lambda)^*,$$

where (4.22) holds for  $\lambda \in \varrho(A_\nu) \cap \varrho(A_{b, \Sigma_+})$ , (4.23) holds for  $\lambda \in \varrho(A_{\chi, \Sigma_+}) \cap \varrho(A_\gamma) \cap \varrho(A_\nu)$ .

**Remark 4.5.** The analysis in Proposition 4.1ff. showed that  $D(A_{\chi, \Sigma_+}) \subset H^{\frac{3}{2}-\varepsilon}(\Omega)$  but is not in general contained in  $H^{\frac{3}{2}}(\Omega)$ . Thus those results in Malamud [M10], Section 6, that concern second-order realizations with domain contained in  $H^{\frac{3}{2}}(\Omega)$  (i.e., with  $\gamma_0 u$  and  $\nu u \in L_2(\Sigma)$ ), will not in general apply to the mixed problem.

**Remark 4.6.** If we instead of (4.9) consider a boundary condition

$$(4.24) \quad \gamma_0 u = g\nu u,$$

where  $g$  is an arbitrary  $C^\infty$ -function on  $\Sigma$ , we can carry an analysis through, showing that if the corresponding realization  $\tilde{A}$  is bijective and selfadjoint, then it corresponds to an operator  $L_1$  from  $H^{-\frac{3}{2}}(\Sigma)$  to  $H^{\frac{3}{2}}(\Sigma)$ , with domain dense in  $H^{-\frac{3}{2}}(\Sigma)$  and acting like  $P_{\nu,\gamma} - g$ , such that there are Krein formulas

$$(4.25) \quad \begin{aligned} \tilde{A}^{-1} - A_\nu^{-1} &= K_\nu L_1^{-1} K_\nu^* = K_\nu (P_{\nu,\gamma} - g)^{-1} K_\nu^*, \\ \tilde{A}^{-1} - A_\gamma^{-1} &= K_\gamma P_{\nu,\gamma} (P_{\nu,\gamma} - g)^{-1} g K_\gamma^*, \end{aligned}$$

and  $\lambda$ -dependent variants. But again, the operator  $L_1^{-1} = (P_{\nu,\gamma} - g)^{-1}$  is nonstandard in the calculus of  $\psi$ do's, since  $P_{\nu,\gamma}$  is elliptic of order  $-1$  whereas  $g$  defines an operator of order  $0$  and can vanish on large subsets of  $\Sigma$ .

#### 4.3 Nonsmooth domains.

We here include some observations on cases where the set  $\Omega$  is not smooth. An interesting variant of the Zaremba problem is where  $\Sigma = \Sigma_+ \cup \Sigma_-$  with  $\Sigma_+$  and  $\Sigma_-$  *meeting at an angle*  $< \pi$ . Then there is the perhaps surprising fact that the realization  $A_{\nu,\Sigma_+}$  of  $-\Delta$  with Dirichlet condition  $\gamma_0 u = 0$  on  $\Sigma_-$  and Neumann condition  $\nu u = 0$  on  $\Sigma_+$  ( $\nu = \gamma_1$ ) can have a *better* regularity than when  $\Omega$  is smooth. Here is an example:

**Example 4.7.** Let  $\Omega'$  be a smooth bounded set that is symmetric in  $x_1$  around  $x_1 = 0$ ; i.e., is preserved under the mapping  $J_1: (x_1, x_2, \dots, x_n) \rightarrow (-x_1, x_2, \dots, x_n)$ . Let  $\Omega = \{x \in \Omega' \mid x_1 > 0\}$ . Then the solutions of the mixed problem for  $-\Delta$  on  $\Omega$  with  $\Sigma_- = \partial\Omega' \cap \{x_1 \geq 0\}$ ,  $\Sigma_+ = \overline{\Omega'} \cap \{x_1 = 0\}$ , are the restrictions to  $\Omega$  of those solutions to the Dirichlet problem for  $\Omega'$  that are invariant under  $J_1$ . (This observation enters in a prominent way in the discussion of isospectral domains for mixed problems by Levitin, Parnowski and Polterovich [LPP06].) Here the domain of the Dirichlet realization of  $-\Delta$  on  $\Omega'$  is in  $H^2(\Omega')$ , hence  $D(A_{\nu,\Sigma_+}) \subset H^2(\Omega)$  (observe that both operators are bijective when defined by the variational construction). In this case  $\Sigma_+$  and  $\Sigma_-$  meet at an angle  $\pi/2$ . — Related results are found for polygonal domains, cf. Grisvard [G85].

More generally, consider the case where  $\Omega$  is such that  $\Sigma_+$  and  $\Sigma_-$  meet at an angle  $< \pi$ , in the way described in Brown [B94]; such domains are by some authors called *creased domains*. It is shown there that the solutions  $u \in H^1(\Omega)$  of

$$(4.26) \quad -\Delta u = 0 \text{ in } \Omega, \quad \gamma_0 u = \varphi \text{ on } \Sigma_-, \quad \gamma_1 u = \psi \text{ on } \Sigma_+,$$

with  $\varphi \in H^1(\Sigma_-^\circ)$ ,  $\psi \in L_2(\Sigma_+)$ , have  $\gamma_0(\nabla u) \in L_2(\Sigma)$ ; in particular  $\gamma_1 u \in L_2(\Sigma)$ . Here  $\Sigma$  just needs to be Lipschitz, in such a way that  $\partial\Sigma_+$  is Lipschitz in  $\Sigma$  (we refer to [B94] for the precise description).

To apply this to  $A_{\nu,\Sigma_+}$ , we restrict to quasi-convex domains  $\Omega$ . They are defined by Gesztesy and Mitrea in [GM11] as a special case of Lipschitz domains including convex domains, which allow showing solvability and regularity theorems for the Dirichlet and Neumann problems for  $-\Delta$  on  $\Omega$  in larger scales of Sobolev-type spaces than in Jerison and Kenig [JK95]; the work builds on Mitrea, Taylor and Vasy [MTV05] and Mazya, Mitrea and Shaposhnikova [MMS10].

**Theorem 4.8.** *Assume that  $\Omega$  is bounded, open and quasi-convex as defined in [GM11]. Assume moreover that  $\Omega$  is creased, in the way that the boundary  $\Sigma$  equals  $\Sigma_+ \cup \Sigma_-$ , where  $\Sigma_+$  and  $\Sigma_-$  meet at an angle  $< \pi$ , as described in [B94]. The realization  $A_{\nu,\Sigma_+}$  of  $-\Delta$  with Neumann condition on  $\Sigma_+$ , Dirichlet condition on  $\Sigma_-$  then has  $D(A_{\nu,\Sigma_+}) \subset H^{\frac{3}{2}}(\Omega)$ .*

*Proof.* We can assume that  $\Sigma_- \neq \emptyset$ . To describe a solution in  $H^1(\Omega)$  of

$$(4.27) \quad -\Delta u = f \text{ in } \Omega, \quad \gamma_0 u = 0 \text{ on } \Sigma_-, \quad \gamma_1 u = 0 \text{ on } \Sigma_+,$$

with  $f \in L_2(\Omega)$ , let  $v$  be the solution of the Dirichlet problem

$$(4.28) \quad -\Delta v = f \text{ in } \Omega, \quad \gamma_0 v = 0 \text{ on } \Sigma;$$

then  $z = u - v$  should be a solution of (4.26) with  $\varphi = 0$ ,  $\psi = -\gamma_1 v|_{\Sigma_+}$ . Since  $v \in H^2(\Omega)$  by [GM11] Th. 10.4,  $\gamma_1 v|_{\Sigma_+} \in H^{\frac{1}{2}}(\Sigma_+^\circ) \subset L_2(\Sigma_+)$ , so the result of Brown [B94] implies that  $\gamma_1 z \in H^1(\Sigma)$ . Then the regularity theorem for the Neumann problem [GM11] Th. 10.8 implies that  $z \in H^{\frac{3}{2}}(\Omega)$ , hence  $u = v + z \in H^{\frac{3}{2}}(\Omega)$ . (Since  $A_{\nu, \Sigma_+}$  is bijective, the solutions we consider are consistent with those considered in [B94].) We conclude that  $D(A_{\nu, \Sigma_+}) \subset H^{\frac{3}{2}}(\Omega)$ .  $\square$

Note the contrast with the informations obtained in Proposition 4.1ff. where  $\Sigma$  is smooth and  $D(A_{\nu, \Sigma_+})$  is in general only in  $H^{\frac{3}{2}-\varepsilon}(\Omega)$ . But that is a case where  $\Sigma_+$  and  $\Sigma_-$  meet at the angle  $\pi$ , which is explicitly excluded in [B94].

The mixed problem in these various forms can, for piecewise smooth domains, be regarded as a special case of crack problems and edge problems, as studied e.g. by Duduchava, Dauge, Costabel, Mazya, Solonnikov and their collaborators, see also Schulze et al. [RS83, HS08]. The results are often described in terms of norms weighted by powers of the distance to the edge; this gives a clarification of the singularities, but can lead outside the Sobolev spaces considered here.

Let us finally mention that for quasi-convex domains there is in [GM11] Th. 10.4 established a homeomorphism  $\hat{\gamma}_D: Z \xrightarrow{\sim} (N^{\frac{1}{2}}(\Sigma))^*$  (generalizing  $\gamma_0$ ), which allows translating formula (2.10) in Section 2 above to a formula like (2.16) with (2.17). Here  $N^{\frac{1}{2}}(\Sigma)$  is a certain Hilbert space related to  $H^{\frac{1}{2}}(\Sigma)$  explained in [GM11], and  $(N^{\frac{1}{2}}(\Sigma))^*$  is its dual space with respect to a sesquilinear duality consistent with the  $L_2(\Sigma)$ -scalar product, such that  $N^{\frac{1}{2}}(\Sigma) \subset L_2(\Sigma) \subset (N^{\frac{1}{2}}(\Sigma))^*$ , with dense, continuous injections. For a general closed realization  $\tilde{A}$  of  $-\Delta$ , let  $X$  and  $Y$  be the closures of  $\hat{\gamma}_D(D(\tilde{A}))$  resp.  $\hat{\gamma}_D(D(\tilde{A}^*))$  in  $(N^{\frac{1}{2}}(\Sigma))^*$ , and let  $V$  resp.  $W$  be their inverse images in  $Z$  (by  $\hat{\gamma}_D^{-1}$ ); in the selfadjoint case,  $Y = X$ . We can then define  $\gamma_V$  to be the restriction of  $\hat{\gamma}_D$  mapping  $V$  homeomorphically to  $X$ ; similarly,  $\gamma_W$  is the restriction of  $\hat{\gamma}_D$  mapping  $W$  homeomorphically to  $Y$ . With this, the considerations in (2.15)–(2.18) are valid, leading to:

**Theorem 4.9.** *When  $\Omega$  is quasi-convex and  $\tilde{A}$  is a general closed realization of  $-\Delta$  with  $0 \in \varrho(\tilde{A})$ , it satisfies the Kreĭn resolvent formula (2.16) with (2.17),  $L$  defined as after (2.15). In particular, for the realization  $A_{\nu, \Sigma_+}$  of the mixed problem, one has with  $X$  = the closure of  $\hat{\gamma}_D(D(A_{\nu, \Sigma_+}))$ ,*

$$(4.29) \quad A_{\nu, \Sigma_+}^{-1} - A_\gamma^{-1} = K_{\gamma, X} L^{-1} (K_{\gamma, X})^*, \quad K_{\gamma, X} = i_V \gamma_V^{-1}: X \rightarrow V \subset H.$$

The rest of Section 2 likewise carries over to the quasi-convex setting, but it must be noted that the Dirichlet-to-Neumann operator is then an abstractly defined operator whose local structure is not so well known. Also  $\lambda$ -dependent variants of (2.16) for realizations of  $-\Delta - \lambda$  are valid when  $\lambda \in \varrho(A_\gamma) \cap \varrho(\tilde{A})$ . The interpretation of the general theory of [G68] for quasi-convex domains is worked out in great detail in [GM11].

We remark however that the Kreĭn formula in [GM11] Th. 16.3 differs from our formula (2.16), particularly when  $X \neq (N^{\frac{1}{2}}(\Sigma))^*$  (which holds for genuine mixed problems).

Upper eigenvalue estimates (1.6) for the resolvent difference (4.29) follow from [B62], cf. Remark 3.3. Asymptotic estimates would demand an effort that to our knowledge has not yet been taken up.

## 5. SPECTRAL ASYMPTOTICS FOR THE MIXED PROBLEM

### 5.1 Notation.

In this section we go back to smooth domains and restrict the attention to the case  $a_{jk} = \delta_{jk}$ , i.e., we take  $A$  principally equal to  $-\Delta$ , in order to use some detailed formulas in Eskin [E81].

We want to show a spectral asymptotic formula for the operator

$$(A_{\chi, \Sigma_+} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1} = K_{\gamma, X}^\lambda (L^\lambda)^{-1} (K_{\gamma, X}^\lambda)^* = -K_{\gamma, X}^\lambda (P_{\gamma, \chi, +}^\lambda)^{-1} (K_{\gamma, X}^\lambda)^*$$



from Theorem 4.2. As done also earlier, we begin by taking  $\lambda$  as a sufficiently low fixed real number such that the considered realizations of  $A - \lambda$  are positive, and then omit  $\lambda$  from the notation. General  $\lambda$  are included in the proof of the final Theorem 5.17.

In view of the formula (2.17) for  $K_{\gamma,X}$ , we are considering the operator

$$(5.1) \quad A_{\chi,\Sigma_+}^{-1} - A_{\gamma}^{-1} = i_V \gamma_V^{-1} L^{-1} (\gamma_V^{-1})^* \text{pr}_V;$$

it is compact selfadjoint nonnegative.

Let us first recall some facts on spaces describing the spectral behavior of compact operators. For  $p > 0$  we denote by  $\mathcal{C}_p$  the Schatten class of compact linear operators  $B$  (in a Hilbert space  $H$ ) with singular value sequences  $(s_j(B))_{j \in \mathbb{N}}$  belonging to  $\ell_p$ , and by  $\mathfrak{S}_p$  the quasi-normed space of compact operators  $B$  with  $s_j(B) = O(j^{-1/p})$  (sometimes called a weak Schatten class); here  $\mathfrak{S}_p \subset \mathcal{C}_{p'}$  for  $p' > p$ . Moreover, we denote by  $\mathfrak{S}_{p,0}$  the subset of operators  $B \in \mathfrak{S}_p$  for which  $s_j(B) = o(j^{-1/p})$ , i.e.,  $s_j(B)j^{1/p} \rightarrow 0$  for  $j \rightarrow \infty$ . Clearly,  $\mathfrak{S}_p \subset \mathfrak{S}_{p',0}$  for  $p' > p$ .

The rules shown by Ky Fan [F51]

$$s_{j+k-1}(B+B') \leq s_j(B) + s_k(B'), \quad s_{j+k-1}(BB') \leq s_j(B)s_k(B'),$$

imply that  $\mathcal{C}_p$ ,  $\mathfrak{S}_p$  and  $\mathfrak{S}_{p,0}$  are vector spaces, and that there are the following product rules:

$$(5.2) \quad \mathcal{C}_p \cdot \mathcal{C}_q \subset \mathcal{C}_{1/(p^{-1}+q^{-1})}, \quad \mathfrak{S}_p \cdot \mathfrak{S}_q \subset \mathfrak{S}_{1/(p^{-1}+q^{-1})}, \quad \mathfrak{S}_p \cdot \mathfrak{S}_{q,0} \subset \mathfrak{S}_{1/(p^{-1}+q^{-1}),0}.$$

Moreover, the rule for  $F_1, F_2 \in \mathcal{L}(H)$ ,

$$(5.3) \quad s_j(F_1 B F_2) \leq \|F_1\| s_j(B) \|F_2\|$$

implies that  $\mathcal{C}_p$ ,  $\mathfrak{S}_p$  and  $\mathfrak{S}_{p,0}$  are preserved under compositions with bounded operators. They are also preserved under taking adjoints. We recall two perturbation results:

**Lemma 5.1.**

1° If  $s_j(B)j^{1/p} \rightarrow C_0$  and  $s_j(B')j^{1/p} \rightarrow 0$  for  $j \rightarrow \infty$ , then  $s_j(B+B')j^{1/p} \rightarrow C_0$  for  $j \rightarrow \infty$ .

2° If  $B = B_M + B'_M$  for each  $M \in \mathbb{N}$ , where  $s_j(B_M)j^{1/p} \rightarrow C_M$  for  $j \rightarrow \infty$  and  $s_j(B'_M)j^{1/p} \leq c_M$  for  $j \in \mathbb{N}$ , with  $C_M \rightarrow C_0$  and  $c_M \rightarrow 0$  for  $M \rightarrow \infty$ , then  $s_j(B)j^{1/p} \rightarrow C_0$  for  $j \rightarrow \infty$ .

The statement in 1° is the Weyl-Ky Fan theorem (cf. e.g. [GK69] Th. II 2.3), and 2° is a refinement shown in [G84], Lemma 4.2.2°.

We also recall that when  $\Xi$  and  $\Xi_1$  are  $m$ -dimensional manifolds (possibly with a boundary, sufficiently smooth),  $\Xi_1$  being compact, and  $B$  is a bounded linear operator from  $L_2(\Xi)$  to  $H^t(\Xi_1)$  for some  $t > 0$ , then  $B \in \mathfrak{S}_{m/t}$  as an operator from  $L_2(\Xi)$  to  $L_2(\Xi_1)$ , with

$$(5.4) \quad s_j(B)j^{t/m} \leq C \|B\|_{\mathcal{L}(L_2(\Xi), H^t(\Xi_1))},$$

with a constant  $C$  depending on  $t$  and the manifolds (references e.g. in [G84]).

**5.2 Constant coefficients.**

One ingredient in the analysis of the spectrum of (5.1) is an application of the constant-coefficient situation, so we begin by working that out, in the case  $b = 0$ . Here  $\Omega$ ,  $\Sigma$  and  $\Sigma_{\pm}$  are replaced by  $\mathbb{R}_+^n$ ,  $\mathbb{R}^{n-1}$  and  $\overline{\mathbb{R}}_{\pm}^{n-1}$ , and we take  $A = -\Delta + \alpha^2$  for some  $\alpha > 0$ ; marking the operators with a subscript 0. The Poisson operator  $K_{0,\gamma}$  solving the Dirichlet problem is the operator  $\varphi(x') \mapsto \mathcal{F}_{\xi' \rightarrow x'}^{-1}[e^{-x_n(|\xi'|^2 + \alpha^2)^{\frac{1}{2}}} \hat{\varphi}(\xi')]$ , so the Dirichlet-to-Neumann operator  $P_{0,\gamma,\nu}$  is the  $\psi$ do with symbol  $-(|\xi'|^2 + \alpha^2)^{\frac{1}{2}}$ , i.e.,

$$P_{0,\gamma,\nu} = -\text{Op}((|\xi'|^2 + \alpha^2)^{\frac{1}{2}}) = -(-\Delta_{x'} + \alpha^2)^{\frac{1}{2}}, \text{ with inverse } P_{0,\nu,\gamma} = -\text{Op}((|\xi'|^2 + \alpha^2)^{-\frac{1}{2}}).$$

( $\mathcal{F}$  denotes the Fourier transform, and  $\text{Op}(a(x', \xi'))v = \mathcal{F}_{\xi' \rightarrow x'}^{-1}(a(x', \xi')\mathcal{F}_{x' \rightarrow \xi'}v)$ .) Then with  $\xi'' = (\xi_1, \dots, \xi_{n-2})$ ,

$$(5.5) \quad L_0 = -r^+ P_{0,\gamma,\nu} e^+ = r^+ \text{Op}((|\xi''|^2 + \xi_{n-1}^2 + \alpha^2)^{\frac{1}{2}}) e^+ : H_0^s(\mathbb{R}_+^{n-1}) \rightarrow H^{s-1}(\mathbb{R}_+^{n-1});$$

it will be used with  $s = 1 - \varepsilon$ , cf. Proposition 4.1. According to Eskin [E81], Ch. 7, one has in view of the factorization

$$(|\xi''|^2 + \xi_{n-1}^2 + \alpha^2)^{\frac{1}{2}} = ((|\xi''|^2 + \alpha^2)^{\frac{1}{2}} - i\xi_{n-1})^{\frac{1}{2}} ((|\xi''|^2 + \alpha^2)^{\frac{1}{2}} + i\xi_{n-1})^{\frac{1}{2}},$$

that  $L_0$  has the inverse

$$(5.6) \quad L_0^{-1} = r^+ \Lambda_+ e^+ r^+ \Lambda_- \ell_s^+ : H^{s-1}(\mathbb{R}_+^{n-1}) \rightarrow H_0^s(\mathbb{R}_+^{n-1}), \quad 0 < s < 1,$$

where

$$(5.7) \quad \Lambda_{\pm} = \text{Op}(\lambda_{\pm}(\xi')), \quad \lambda_{\pm}(\xi') = ((|\xi''|^2 + \alpha^2)^{\frac{1}{2}} \pm i\xi_{n-1})^{-\frac{1}{2}},$$

and  $\ell_s^+$  denotes a smooth extension operator, continuous from  $H^t(\mathbb{R}_+^{n-1})$  to  $H^t(\mathbb{R}^{n-1})$  for all  $t$ . The operators  $\Lambda_{\pm}$  are a “plus-operator” resp. a “minus-operator” in the terminology of [E81]; plus-operators preserve support in  $\overline{\mathbb{R}_+^{n-1}}$ , and minus-operators are adjoints of plus-operators and preserve support in  $\overline{\mathbb{R}_-^{n-1}}$ .

When the formula is used for  $s = 1 - \varepsilon > \frac{1}{2}$ , we can replace  $\ell_s^+$  by  $e^+$ , so

$$(5.8) \quad L_0^{-1} = r^+ \Lambda_+ e^+ r^+ \Lambda_- e^+ = \Lambda_{+,+} \Lambda_{-,+} : H^{-\varepsilon}(\mathbb{R}_+^{n-1}) \rightarrow H_0^{1-\varepsilon}(\mathbb{R}_+^{n-1})$$

(recall the notation  $Q_+ = r^+ Q e^+$ ).  $L_0^{-1}$  is of course different from  $(\Lambda_+ \Lambda_-)_+ = -P_{0,\nu,\gamma,+}$ , that we shall compare it with further below. We note that  $\Lambda_{-,+}$  maps  $H^{-\varepsilon}(\mathbb{R}_+^{n-1})$  to  $H^{\frac{1}{2}-\varepsilon}(\mathbb{R}_+^{n-1}) = H_0^{\frac{1}{2}-\varepsilon}(\mathbb{R}_+^{n-1})$ . Then the fact that  $\Lambda_{+,+}$  preserves support in  $\overline{\mathbb{R}_+^{n-1}}$ , confirms that the range of  $L^{-1}$  is in the subspace  $H_0^{1-\varepsilon}(\mathbb{R}_+^{n-1})$  of  $H^{1-\varepsilon}(\mathbb{R}_+^{n-1})$ .

We shall treat our general problem by reducing to cases in local coordinates with ingredients principally of this form. Then  $L_0^{-1}$  is multiplied on both sides with cutoff functions, so we shall now also consider  $\psi L_0^{-1} \psi_1$ , where  $\psi, \psi_1 \in C_0^\infty(B_R)$  for some ball  $B_R = \{|x'| < R\} \subset \mathbb{R}^{n-1}$ . It is continuous

$$(5.9) \quad \psi L_0^{-1} \psi_1 : L_2(B_R \cap \mathbb{R}_+^{n-1}) \rightarrow H^{1-\varepsilon}(B_R \cap \mathbb{R}_+^{n-1}), \quad \text{any } \varepsilon > 0;$$

hence in view of (5.4),

$$(5.10) \quad \psi L_0^{-1} \psi_1 \in \mathfrak{S}_{n-1+\delta}, \quad \text{any } \delta > 0.$$

(Better estimates will be obtained below.) We shall compare it with  $-\psi r^+ P_{0,\gamma,\nu}^{-1} e^+ \psi_1 = -\psi P_{0,\nu,\gamma,+} \psi_1$ , and for this purpose we observe that

$$\begin{aligned} -r^+ P_{0,\nu,\gamma} e^+ - L^{-1} &= r^+ \Lambda_+ \Lambda_- e^+ - r^+ \Lambda_+ e^+ r^+ \Lambda_- e^+ \\ &= r^+ \Lambda_+ e^- J J r^- \Lambda_- e^+ = G^+(\Lambda_+) G^-(\Lambda_-), \end{aligned}$$

where  $r^-$  is the restriction operator from  $\mathbb{R}^{n-1}$  to  $\mathbb{R}_-^{n-1}$ ,  $e^-$  is the corresponding extension-by-zero operator, and  $J$  is the reflection operator  $J: u(x'', x_{n-1}) \mapsto u(x'', -x_{n-1})$ . We have used that  $I - e^+ r^+ = e^- r^-$ , and denoted

$$(5.11) \quad G^+(Q) = r^+ Q e^- J, \quad G^-(Q) = J r^- Q e^+,$$

as in [G84] and subsequent papers and books of the author. Note that the distribution kernel of  $G^+(\Lambda_+)$  is obtained from that of  $\Lambda_+$  by restriction to the second quadrant in  $(y_{n-1}, x_{n-1})$ -space, so that the singularity at the diagonal  $\{x_{n-1} = y_{n-1}\}$  is only felt at 0.

On the manifold  $\Sigma = \Sigma_+ \cup \Sigma_-$ ,  $G^\pm(Q)$  make sense only in local coordinates, but

$$(5.12) \quad L(Q_1, Q_2) = (Q_1 Q_2)_+ - Q_{1,+} Q_{2,+}$$

is well defined when  $Q_1$  and  $Q_2$  are of order  $\leq 0$ , and locally has the structure  $G^+(Q_1) G^-(Q_2)$ .

For later purposes we recall the result of Laptev [L81] (also shown for  $\psi$ do's having the transmission property in [G84]):

**Theorem 5.2.** [L81] *Let  $n-1 \geq 2$ . When  $Q$  is a  $\psi$ do on  $\mathbb{R}^{n-1}$  of order  $-r < 0$ , and  $\psi \in C_0^\infty(\mathbb{R}^{n-1})$ , then  $\psi G^\pm(Q)$  and  $G^\pm(Q)\psi$  are in  $\mathfrak{S}_{(n-2)/r}$ , with  $s_j j^{r/(n-2)}$  converging to a limit determined from the principal symbol.*

*When  $Q_1$  and  $Q_2$  are  $\psi$ do's on  $\Sigma = \Sigma_+ \cup \Sigma_-$  of orders  $-r_1, -r_2 < 0$ , then  $L(Q_1, Q_2)$  is in  $\mathfrak{S}_{(n-2)/(r_1+r_2)}$ .*

The operators  $\Lambda_\pm$  are of order  $-\frac{1}{2}$ , but are not standard  $\psi$ do's, since the symbols  $\lambda_\pm$  are not in Hörmander's symbol space  $S_{1,0}^{-\frac{1}{2}}$  as functions of  $\xi'$  (high derivatives in  $\xi''$  do not satisfy the required estimates in terms of powers of  $1 + |\xi'|$ ). Then Laptev's theorem is not applicable to  $G^\pm(\Lambda_+)$  and  $G^\pm(\Lambda_-)$ . In fact, one can check that the associated integral operator kernels, calculated explicitly, do not satisfy all the estimates required for Th. 3 in [L81]. We expect that it should be possible to show a spectral estimate as in Theorem 5.2 for these operators, but leave out further investigations here, settling for some weaker estimates that still serve our purpose.

In the following, we denote  $x_{n-1} = t$ ,  $y_{n-1} = s$ , with dual variables  $\tau, \sigma$ , to simplify the notation. Let  $\zeta(t) \in C^\infty(\mathbb{R})$ , taking values in  $[0, 1]$  and equal to 1 for  $t \geq 1$ , equal to 0 for  $t \leq \frac{2}{3}$ . For  $\varepsilon > 0$ , denote  $\zeta(t/\varepsilon) = \zeta_\varepsilon(t)$ .

**Lemma 5.3.** *Let  $\varepsilon > 0$ . The operators  $\zeta_\varepsilon G^+(\Lambda_\pm)$  are of order  $-\frac{3}{2}$ , and  $\psi \zeta_\varepsilon G^+(\Lambda_\pm)$  as well as  $\zeta_\varepsilon G^+(\Lambda_\pm)\psi$  belong to  $\mathfrak{S}_{2(n-1)/3} \cup \mathcal{C}_1$ . Similarly,  $G^-(\Lambda_\pm)\zeta_\varepsilon$  are of order  $-\frac{3}{2}$ , and  $G^-(\Lambda_\pm)\zeta_\varepsilon\psi$ ,  $\psi G^-(\Lambda_\pm)\zeta_\varepsilon$  belong to  $\mathfrak{S}_{2(n-1)/3} \cup \mathcal{C}_1$ .*

*Proof.* It suffices to give the details for  $\varepsilon = 1$ . Consider  $G^+(\Lambda_+)$ . First we note that

$$\zeta G^+(\Lambda_+) = \zeta r^+ \Lambda_+ e^- J = r^+ \Lambda_+ \zeta e^- J + r^+ [\zeta, \Lambda_+] e^- J = r^+ [\zeta, \Lambda_+] e^- J,$$

since  $\zeta e^- = 0$ ; here  $[\zeta, \Lambda_+]$  is the commutator  $\zeta \Lambda_+ - \Lambda_+ \zeta$ . As for ordinary  $\psi$ do's, the commutator is of lower order; since  $\Lambda_+$  is nonstandard, we work out proof details:

For  $t, s \in \mathbb{R}$ ,  $\zeta$  has the Taylor-expansion

$$\begin{aligned} \zeta(t) &= \sum_{0 \leq j < J} \frac{1}{j!} \zeta^{(j)}(s) (t-s)^j + (t-s)^J \varrho_J(s, t), \text{ where} \\ \varrho_J(s, t) &= \frac{1}{(J-1)!} \int_0^1 (1-h)^{J-1} \partial^J \zeta(s+h(t-s)) dh. \end{aligned}$$

Then, using that  $(t-s)^j e^{i(t-s)\tau} = D_\tau^j e^{i(t-s)\tau}$  and integrating by parts (as allowed in oscillatory integrals), we find, denoting  $(2\pi)^{1-n} d\xi' = d\xi'$ :

$$\begin{aligned} (5.13) \quad [\zeta, \Lambda_+] u &= \int e^{i(x'-y') \cdot \xi'} (\zeta(t) - \zeta(s)) \lambda_+(\xi') u(y') d\xi' dy' \\ &= \int e^{i(x'-y') \cdot \xi'} \left( \sum_{1 \leq j < J} \frac{1}{j!} \zeta^{(j)}(s) (t-s)^j + (t-s)^J \varrho_J(s, t) \right) \lambda_+(\xi') u(y') d\xi' dy' \\ &= \int e^{i(x'-y') \cdot \xi'} \left( \sum_{1 \leq j < J} \frac{1}{j!} \zeta^{(j)}(s) \overline{D}_\tau^j + \varrho_J(s, t) \overline{D}_\tau^J \right) \lambda_+(\xi') u(y') d\xi' dy' \\ &= \sum_{1 \leq j < J} \frac{1}{j!} \text{Op}(\overline{D}_\tau^j \lambda_+(\xi')) \zeta^{(j)} u + \text{Op}(\varrho_J(s, t) \overline{D}_\tau^J \lambda_+(\xi')) u. \end{aligned}$$

Here

$$\overline{D}_\tau^j \lambda_+(\xi') = \overline{D}_\tau^j (|(\xi'', \alpha)| + i\tau)^{-\frac{1}{2}} = c_j (|(\xi'', \alpha)| + i\tau)^{-\frac{1}{2}-j};$$

they are of order  $-\frac{1}{2} - j$ . Take  $J$  so large that the last symbol is integrable in  $\xi'$ , e.g.  $J = n$ . Then the terms in the sum over  $j$  map  $H^r(\mathbb{R}^{n-1})$  into  $H^{r+\frac{1}{2}+j}(\mathbb{R}^{n-1})$  ( $r \in \mathbb{R}$ ) by elementary considerations, and the last term has a continuous kernel, supported for  $s, t \in [\frac{1}{3}, \frac{4}{3}]$ . Similar considerations hold

for  $\zeta G^+(\Lambda_-)$ . When we cut down with multiplication by  $\psi$ , and functions  $1_{\mathbb{R}_\pm^{n-1}}$ , we can use the spectral estimates (5.4) and the trace-class property of operators with continuous kernel, to see that  $\psi \zeta G^+(\Lambda_\pm)$  are in  $\mathfrak{S}_{2(n-1)/3} \cup \mathcal{C}_1$ .

The statements for  $G^-(\Lambda_\pm)\zeta$  are shown similarly, and for the operators with  $\psi$  to the right one can use that  $G^+(\Lambda_\pm)$  and  $G^-(\Lambda_\mp)$  are adjoints.  $\square$

One could argue in a more refined way (e.g. with sequences of nested cutoff functions), to show that since  $\zeta$  is supported away from 0, *all* the terms in (5.13) give spectrally negligible contributions (in  $\bigcap_p \mathfrak{S}_p$ ) when we take  $G^+$  of them (as for ordinary singular Green operators), but that extra information will not be needed in the following.

Next, we shall show spectral estimates of the contributions to  $G^\pm(\Lambda_\pm)$  supported near  $t = 0$ . Here we shall profit from the fact that Birman and Solomyak in [BS77] showed far-reaching spectral results for nonstandard  $\psi$ do's, taking  $L_p$ -norms (not just  $L_\infty$ -norms) of cutoff functions into account. Anisotropic symbols are allowed there, but we just need the case of isotropic symbols with low smoothness.

**Theorem 5.4.** [BS77] *Let  $A = \text{Op}(b(x)a(x, \xi)c(y))$  on  $\mathbb{R}^m$ , with  $a(x, \xi)$  homogeneous in  $\xi$  of degree  $-\mu \in ]-m, 0[$ . Denote  $m/\mu = \nu$ . Then  $A \in \mathfrak{S}_\nu$  with*

$$\sup_{j \in \mathbb{N}} s_j(A) j^{1/\nu} \leq C \|b\|_{L_{q_1}} \|c\|_{L_{q_2}} [a_{|\xi|=1}]_\beta,$$

if

$$q_1, q_2 \in ]2, \infty], \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{\nu}, \quad \beta = q_1.$$

(Here  $[\Phi(x, \xi)]_\beta$  denotes the norm of a certain linear operator on  $\mathfrak{S}_\beta$  defined from  $\Phi$ .) A sufficient condition for the boundedness of  $[a_{|\xi|=1}]_\beta$  is that

$$(5.14) \quad a(x, \xi)|_{|\xi|=1} \in L_\infty(S_\xi^{m-1}, W_p^l(\mathbb{R}_x^m)), \quad \text{with } \frac{1}{2} - \frac{1}{q_1} < \frac{1}{p} \leq \frac{1}{2}, \quad pl > m.$$

The paper [BS77] also covers cases where  $\nu \leq 1$ , and gives spectral asymptotics formulas under additional mild regularity hypotheses (in (5.14),  $L_\infty$  is then replaced by  $C^0$ ).

In order to apply the result we must estimate the effect of replacing the  $\Lambda_\pm(\alpha)$  by the operators  $\Lambda_\pm(0)$  with strictly homogeneous symbols  $\lambda_\pm(\xi', 0)$ .

**Lemma 5.5.** *The symbol  $\lambda_\pm(\xi', \alpha) - \lambda_\pm(\xi', 0)$  of the difference  $\Lambda_\pm(\alpha) - \Lambda_\pm(0)$  satisfies*

$$\lambda_\pm(\xi', \alpha) - \lambda_\pm(\xi', 0) = O(\alpha^2 |(\xi', \alpha)|^{-\frac{3}{2}} |\xi'|^{-1} |(\xi'', \alpha)|^{-1}).$$

Hence it defines an operator mapping  $H^r(\mathbb{R}^{n-1})$  into  $H_{\text{loc}}^{r+\frac{5}{2}}(\mathbb{R}^{n-1})$  for  $r \in \mathbb{R}$ .

It follows that  $\psi(\Lambda_+(\alpha) - \Lambda_+(0))$ ,  $\psi(G^+(\Lambda_+(\alpha)) - G^+(\Lambda_+(0)))$ ,  $(\Lambda_-(\alpha) - \Lambda_-(0))\psi_1$  and  $(G^-(\Lambda_-(\alpha)) - G^-(\Lambda_-(0)))\psi_1$  are in  $\mathfrak{S}_{2(n-1)/5}$ .

*Proof.* We give the details for  $\Lambda_+(\alpha) - \Lambda_+(0)$ . Here

$$\begin{aligned} \lambda_+(\xi', \alpha) - \lambda_+(\xi', 0) &= \frac{\lambda_+(\xi', \alpha)^2 - \lambda_+(\xi', 0)^2}{\lambda_+(\xi', \alpha) + \lambda_+(\xi', 0)} = \frac{1}{\lambda_+(\xi', \alpha) + \lambda_+(\xi', 0)} \left( \frac{1}{|(\xi'', \alpha)| + i\tau} - \frac{1}{|\xi''| + i\tau} \right) \\ &= \frac{|\xi''| - |(\xi'', \alpha)|}{(\lambda_+(\xi', \alpha) + \lambda_+(\xi', 0))(|(\xi'', \alpha)| + i\tau)(|\xi''| + i\tau)} \\ &= \frac{-\alpha^2}{(\lambda_+(\xi', \alpha) + \lambda_+(\xi', 0))(|(\xi'', \alpha)| + i\tau)(|\xi''| + i\tau)(|\xi''| + |(\xi'', \alpha)|)} \\ &= O(\alpha^2 |(\xi', \alpha)|^{-\frac{3}{2}} |\xi'|^{-1} |(\xi'', \alpha)|^{-1}). \end{aligned}$$

The operator with symbol  $\zeta(|\xi'|)(\lambda_+(\xi', \alpha) - \lambda_+(\xi', 0))$  maps  $H^r(\mathbb{R}^{n-1})$  into  $H^{r+\frac{5}{2}}(\mathbb{R}^{n-1})$  for  $r \in \mathbb{R}$ , and the remainder supported near  $|\xi'| = 0$  gives an operator mapping into  $C^\infty(\mathbb{R}^{n-1})$ . When cutoffs by compactly supported functions are applied, this gives operators in  $\mathfrak{S}_{2(n-1)/5}$ .

The result for  $\Lambda_-$  follows by similar calculations or by duality.  $\square$

In the following,  $\varphi(t)$  denotes a function in  $C^\infty(\mathbb{R})$  that takes values in  $[0, 1]$  and equals 1 for  $|t| \leq \frac{1}{3}$ , equals 0 for  $|t| \geq \frac{2}{3}$ ; we denote  $\varphi(t/\varepsilon) = \varphi_\varepsilon(t)$ . We can assume that  $1_{\mathbb{R}_+}(1 - \zeta) = 1_{\mathbb{R}_+}\varphi$ .

**Lemma 5.6.** *There are the following spectral estimates:*

$$\begin{aligned}
 (5.15) \quad & \sup_j s_j(\varphi_\varepsilon(t)\psi\Lambda_+(0))j^{1/(2n-2)} \leq C_\varepsilon, \\
 & \sup_j s_j(\varphi_\varepsilon(t)\psi G^+(\Lambda_+(0)))j^{1/(2n-2)} \leq C_\varepsilon, \\
 & \sup_j s_j(\Lambda_-(0)\psi_1\varphi_\varepsilon(t))j^{1/(2n-2)} \leq C_\varepsilon, \\
 & \sup_j s_j(G^-(\Lambda_-(0))\psi_1\varphi_\varepsilon(t))j^{1/(2n-2)} \leq C_\varepsilon.
 \end{aligned}$$

where  $C_\varepsilon \rightarrow 0$  for  $\varepsilon \rightarrow 0$ .

*Proof.* For the first line in (5.15), we apply Theorem 5.4 with

$$b(x') = \varphi_\varepsilon(t)\psi_2(x'), \quad a(x', \xi') = \psi(x')\lambda_+(\xi', 0), \quad c(x') = 1,$$

where  $\psi_2 \in C_0^\infty(\mathbb{R}^{n-1})$ , equal to 1 on  $\text{supp } \psi$ . Here  $m = n - 1$ ,  $\mu = \frac{1}{2}$  so that  $\nu = m/\mu = 2(n - 1)$ , and we take  $q_1 = \beta = \nu = 2(n - 1)$  and  $q_2 = \infty$ . Moreover, since  $\frac{1}{2} - \frac{1}{q_1} = \frac{1}{2} - \frac{1}{2(n-1)} = \frac{n-2}{2(n-1)}$ ,  $p$  is taken in  $]2, \frac{2(n-1)}{n-2}]$  and  $l$  is taken  $> (n - 1)/p$ . (5.14) is satisfied since  $\psi \in C_0^\infty(\mathbb{R}^{n-1}) \subset W_p^l(\mathbb{R}^{n-1})$ . Then

$$\begin{aligned}
 \sup_j s_j(\varphi_\varepsilon(t)\psi\Lambda_+)j^{1/(2n-2)} &\leq C\|\varphi_\varepsilon\psi_2\|_{L_\nu} \leq C' \text{vol}(\text{supp}(\varphi_\varepsilon\psi_2))^{1/(2n-2)} \\
 &\leq C'' \varepsilon^{1/(2n-2)} \rightarrow 0
 \end{aligned}$$

for  $\varepsilon \rightarrow 0$ .

For the second line in (5.15) we replace  $b$  by  $1_{\mathbb{R}_+^{n-1}}\varphi_\varepsilon\psi_2$  and  $c$  by  $1_{\mathbb{R}_-^{n-1}}$ , and use that  $J$  is an isometric isomorphism.

The proof of the third and fourth line goes in a similar way, interchanging choices for  $b$  and  $c$ .  $\square$

We can finally conclude:

**Theorem 5.7.** *The operator  $L_0^{-1}$  acts like*

$$(5.16) \quad L_0^{-1} = -P_{0,\nu,\gamma,+} - G^+(\Lambda_+)G^-(\Lambda_-),$$

where

$$(5.17) \quad P_{0,\nu,\gamma,+}: H^s(\mathbb{R}_+^{n-1}) \rightarrow H^{s+1}(\mathbb{R}_+^{n-1}) \text{ for } -\frac{1}{2} < s < \frac{1}{2},$$

and the operators  $\psi G^\pm(\Lambda_\pm)$  and  $G^\pm(\Lambda_\pm)\psi$  are in  $\mathfrak{S}_{2(n-1),0}$ , when  $\psi \in C_0^\infty(\mathbb{R}^{n-1})$ .

*Proof.* The decomposition (5.16) was shown above. The continuity in (5.17) follows since  $P_{0,\nu,\gamma}$  is a constant-coefficient  $\psi$ do of order  $-1$ . For the next statement, we give details for  $\psi G^+(\Lambda_+)$ ; the other cases are similar. For any  $\varepsilon > 0$  we can write

$$(5.18) \quad \psi G^+(\Lambda_+(\alpha)) = \varphi_\varepsilon(t)\psi G^+(\Lambda_+(0)) + \varphi_\varepsilon(t)\psi[G^+(\Lambda_+(\alpha)) - G^+(\Lambda_+(0))] + \zeta_\varepsilon(t)\psi G^+(\Lambda_+(\alpha)).$$

Here the first term satisfies (5.15), the second term is in  $\mathfrak{S}_{2(n-1)/5}$  by Lemma 5.5, and the third term is in  $\mathfrak{S}_{\max\{2(n-1)/3, 1+\delta\}}$  (for any  $\delta > 0$ ) by Lemma 5.3. Thus the sum of the second and third term satisfies  $s_j j^{1/(2n-2)} \rightarrow 0$  for  $j \rightarrow \infty$ . We can then apply Lemma 5.1 2°, with  $1/p = 1/(2n - 2)$ ,  $M = 1/\varepsilon$ ,  $B_M$  being the sum of the second and third terms and  $B'_M$  being the first term,  $c_M = C_\varepsilon$  and  $C_M = C_0 = 0$ .  $\square$

**Remark 5.8.** In the case  $n = 2$ , when  $Q$  is a  $\psi$ do on  $\mathbb{R}^{n-1}$  of order  $-r < 0$ , the operators  $G^\pm(Q)$  are not covered by Theorem 5.2. But certainly the calculations leading to Theorem 5.7 work in this case, so we have at least that  $\psi G^\pm(Q) \in \mathfrak{S}_{(n-1)/r, 0}$ . Similarly, if  $n = 2$  and  $Q_1$  and  $Q_2$  are  $\psi$ do's on  $\Sigma$  of negative orders  $-r_1, -r_2$ , then  $L(Q_1, Q_2) \in \mathfrak{S}_{(n-1)/(r_1+r_2)}$ .

### 5.3 Variable coefficients, analysis of $L^{-1}$ .

Now consider  $A = -\Delta + a_0(x)$  on the smooth bounded open subset  $\Omega$  of  $\mathbb{R}^n$ , provided with the mixed boundary condition  $\nu u = b\gamma_0 u$  on  $\Sigma_+$ ,  $\gamma_0 u = 0$  on  $\Sigma_-$ . The operator  $L$  acts like

$$L\varphi = r^+(b - P_{\gamma, \nu})e^+\varphi = -P_{\gamma, \chi, +}\varphi$$

for  $\varphi \in D(L)$ , cf. (4.3), (4.6).

In the analysis of  $L^{-1}$  on  $\Sigma_+$  we want to use the insight gained in Section 5.2 for the “flat” constant-coefficient case, but since the ingredients are not standard  $\psi$ do's, we do not have the usual localization tools for  $\psi$ do's available and must reason very carefully (for example in Eskin's book, formulas for coordinate changes are only worked out for a subclass of symbols with better estimates than the present  $\lambda_\pm(\xi')$ ). The strategy will be to reduce to a situation where the results from the “flat” case can be used directly.

Our aim is to show:

**Theorem 5.9.** *The operator  $L^{-1}$  acts like  $-P_{\nu, \gamma, +} + R$ , where  $R \in \mathfrak{S}_{n-1, 0}$ . In particular,  $L^{-1} \in \mathfrak{S}_{n-1}$ .*

This will be shown in several steps. We first show a preliminary spectral estimate for  $L^{-1}$ ; it will be improved later.

**Lemma 5.10.** *The operator  $L^{-1}: X^* \rightarrow X$  extends to an operator  $M$  that maps continuously*

$$(5.19) \quad M: H^s(\Sigma_+^\circ) \rightarrow H_0^{s+\frac{1}{2}-\varepsilon}(\Sigma_+) \text{ for } -1 < s \leq \frac{1}{2}.$$

*In particular, the closure of  $L^{-1}$  in  $L_2(\Sigma_+)$  is a continuous operator from  $L_2(\Sigma_+)$  to  $H_0^{\frac{1}{2}-\varepsilon}(\Sigma_+)$ ; it belongs to  $\mathfrak{S}_{(n-1)/(\frac{1}{2}-\varepsilon)}$  for  $\varepsilon > 0$ .*

*Proof.* It follows from Proposition 4.1 that  $L^{-1}$  is continuous from  $X^* = H^{\frac{1}{2}}(\Sigma_+^\circ)$  to  $H_0^{1-\varepsilon}(\Sigma_+)$ . Then it has an adjoint  $M$  (with respect to dualities consistent with the  $L_2(\Sigma_+)$ -scalar product) that is continuous from  $H^{-1+\varepsilon}(\Sigma_+^\circ)$  to  $H_0^{-\frac{1}{2}}(\Sigma_+)$ . But since  $L^{-1}$  is known to be selfadjoint (from  $X^*$  to  $X$ , consistently with the  $L_2$ -scalar product),  $M$  must be an extension of  $L^{-1}$ . Now (5.19) follows by interpolation. For  $s = 0$  we find the last statement, where the spectral information follows from (5.4); note that  $H_0^{\frac{1}{2}-\varepsilon}(\Sigma_+) = H^{\frac{1}{2}-\varepsilon}(\Sigma_+^\circ)$ .  $\square$

When  $\{\varrho_1, \dots, \varrho_N\}$  is any partition of unity for  $\Sigma$ , then  $L^{-1} = \sum_{k=1}^N \varrho_k L^{-1}$ , and it suffices to analyze the terms  $\varrho_k L^{-1}$  individually. Here we can also introduce a cutoff function  $\psi_k$  to the right, considering terms  $\varrho_k L^{-1} \psi_k$  where  $\psi_k$  is 1 on the support of  $\varrho_k$ ; the effect of such a modification will be studied later.

Our next observation is that it is allowed to perform smooth diffeomorphisms of  $\Omega$ , in particular of  $\Sigma$ . Assume that  $\kappa$  is a diffeomorphism of an open neighborhood  $U_0$  of  $\overline{\Omega}$  onto another open set  $V_0 \subset \mathbb{R}^n$ , where  $\kappa(\Omega) = \Omega'$ , then functions  $f(x)$  on  $\Omega$  are carried over to functions  $\underline{f}(y) = f(\kappa^{-1}(y))$  on  $\Omega'$ , and operators  $P$  over  $\overline{\Omega}$  are carried over to operators  $\underline{P}$  over  $\overline{\Omega'}$ :

$$(5.20) \quad (\underline{P}\underline{f})(y) = (Pf)(\kappa^{-1}(y)).$$

The  $\psi$ do  $P_{\gamma, \chi}$  on  $\Sigma$  carries over to a  $\psi$ do  $\underline{P}_{\gamma, \chi}$  on  $\underline{\Sigma}$  according to well-known rules; it is again elliptic of order 1 and has the same principal symbol. The operator  $L$  carries over to  $\underline{L}$ , equal to the truncated

version of  $\underline{P}_{\gamma, \chi}$ , where we apply  $e^+$  and  $r^+$  with respect to the partition  $\underline{\Sigma} = \underline{\Sigma}_+ \cup \underline{\Sigma}_-$ . There is again an inverse  $\underline{L}^{-1}$ , with mapping properties as explained for  $L^{-1}$ , relative to the transformed sets.

To find the structure of  $L^{-1}$  in a neighborhood of a point  $x_0 \in \Sigma$ , let us consider  $\psi L^{-1} \psi_1$ , where  $\psi$  and  $\psi_1$  are  $C^\infty$ -functions supported in the neighborhood, with  $\psi_1 = 1$  on  $\text{supp } \psi$ .

It can be assumed, after a translation and rotation if necessary, that  $x_0 \in \overline{\mathbb{R}}_+^{n-1}$  and the interior normal at  $x_0 = \{x_{0,1}, \dots, x_{0,n-1}, 0\}$  is  $(0, \dots, 0, 1)$ , such that  $x_{0,n-1} > 0$  if  $x_0 \in \Sigma_+^\circ$  and  $x_0 = 0$  if  $x_0 \in \partial\Sigma_+$ ; in the latter case we can assume that the interior normal to  $\partial\Sigma_+ \subset \Sigma$  at  $x_0$  is  $\{0, \dots, 0, 1, 0\}$ . We choose a diffeomorphism that changes  $\overline{\Omega}$  only near  $x_0$ . If  $x_0 \in \Sigma_+^\circ$ , we can assume that  $\psi$  and  $\psi_1$  are supported away from  $\partial\Sigma_+$ ; then we let the diffeomorphism be such that it transforms a neighborhood  $U \subset \mathbb{R}^n$  of  $x_0$  over to  $V \subset \mathbb{R}^n$ , carrying  $U \cap \Omega$  and  $U \cap \Sigma_+$  over to  $V \cap \mathbb{R}_+^n$  and  $V \cap \overline{\mathbb{R}}_+^{n-1}$ , with  $\psi$  and  $\psi_1$  supported in  $U \cap \Sigma_+^\circ$ . If  $x_0 \in \partial\Sigma_+$ , we choose the diffeomorphism such that  $U \cap \Omega$ ,  $U \cap \Sigma$  and  $U \cap \Sigma_+$  are mapped to  $V \cap \mathbb{R}^n p$ ,  $V \cap \mathbb{R}^{n-1}$  and  $V \cap \overline{\mathbb{R}}_+^{n-1}$ ,  $\psi$  and  $\psi_1$  supported in  $V \cap \mathbb{R}^{n-1}$ . (The identifications of  $\mathbb{R}^{n-1}$  and  $\overline{\mathbb{R}}_+^{n-1}$  with  $\mathbb{R}^{n-1} \times \{0\}$  and  $\overline{\mathbb{R}}_+^{n-1} \times \{0\}$  as subsets of  $\mathbb{R}^n$  are understood here.)

This gives a transformed operator  $\underline{\psi} \underline{L}^{-1} \underline{\psi}_1$  acting on functions supported in  $V' = V \cap \mathbb{R}^{n-1}$ . For simplicity of notation, we drop the underlines in the following.

We shall compare  $\psi L^{-1} \psi_1$  with  $\psi L_0^{-1} \psi_1$  where  $L_0^{-1}$  is the constant-coefficient operator studied in Section 5.2. Let us give the details for the most delicate case  $x_0 \in \partial\Sigma_+$ , where the effects of truncation have to be taken into account.

**Proposition 5.11.** *In the setting described in the preceding lines, we have that*

$$(5.21) \quad \psi L^{-1} \psi_1 = -\psi P_{\nu, \gamma, +} \psi_1 + R_1,$$

as operators in  $L_2(V' \cap \mathbb{R}_+^{n-1})$ , where  $R_1 \in \mathfrak{S}_{n-1,0}$ .

*Proof.* It follows from Theorem 5.7 that

$$(5.22) \quad \psi L_0^{-1} \psi_1 = -\psi P_{0, \nu, \gamma, +} \psi_1 + R_2,$$

where  $R_2 = \psi G^+(\Lambda_+) G^-(\Lambda_-) \psi_1$  is in  $\mathfrak{S}_{n-1,0}$ ; cf. (5.2). We shall now compare  $\psi L^{-1} \psi_1$  and  $\psi L_0^{-1} \psi_1$ . There is the difficulty that the operators  $L_1^{-1}$  and  $L_0^{-1}$  do not act over the same manifold, but this will be dealt with by introduction of more cutoff functions. Let  $\psi_2 \in C_0^\infty(V')$ , satisfying  $\psi_2 = 1$  on  $\text{supp } \psi_1$ . We calculate:

$$(5.23) \quad \begin{aligned} \psi L^{-1} \psi_1 - \psi L_0^{-1} \psi_1 &= \psi L^{-1} \psi_2 \psi_1 - \psi \psi_2 L_0^{-1} \psi_1 \\ &= \psi L^{-1} \psi_2 L_0 L_0^{-1} \psi_1 - \psi L^{-1} L \psi_2 L_0^{-1} \psi_1. \end{aligned}$$

We want to insert the factor  $\psi_2$  in the middle of  $L_0 L_0^{-1}$  as well as  $L^{-1} L$ ; this is justified as follows: Write e.g.

$$\psi L^{-1} \psi_2 L_0 L_0^{-1} \psi_1 = \psi L^{-1} \psi_2 L_0 \psi_2 L_0^{-1} \psi_1 + \psi L^{-1} \psi_2 L_0 (1 - \psi_2) L_0^{-1} \psi_1.$$

For the last term, we note that (since  $(1 - \psi_2) \psi_1 = 0$ )

$$(1 - \psi_2) L_0^{-1} \psi_1 = [(1 - \psi_2), L_0^{-1}] \psi_1 = [L_0^{-1}, \psi_2] \psi_1 = L_0^{-1} [\psi_2, L_0] L_0^{-1} \psi_1,$$

where  $[L_0, \psi_2] = [-P_{0, \gamma, \nu, +}, \psi_2]$  is  $L_2$ -bounded (since  $P_{0, \gamma, \nu}$  is a first-order  $\psi$ do). Then

$$\psi L^{-1} \psi_2 L_0 (1 - \psi_2) L_0^{-1} \psi_1 = \psi L^{-1} \psi_2 L_0 L_0^{-1} [\psi_2, L_0] L_0^{-1} \psi_1 = \psi L^{-1} \psi_2 [\psi_2, L_0] L_0^{-1} \psi_1,$$

which is the composition of  $\psi L^{-1} \psi_2 \in \mathfrak{S}_{(n-1)/(\frac{1}{2}-\varepsilon)}$  (cf. Lemma 5.10), the bounded operator  $[\psi_2, L_0]$ , and  $L_0^{-1} \psi_1 \in \mathfrak{S}_{n-1}$  (cf. Theorem 5.7; its adjoint is  $\overline{\psi}_1 L_0^{-1}$ ). Then the whole term is in  $\mathfrak{S}_{(n-1)/(\frac{3}{2}-\varepsilon)}$ , and

$$\psi L^{-1} \psi_2 L_0 L_0^{-1} \psi_1 = \psi L^{-1} \psi_2 L_0 \psi_2 L_0^{-1} \psi_1 + R_3, \text{ where } R_3 \in \mathfrak{S}_{(n-1)/(\frac{3}{2}-\varepsilon)}.$$

Similarly, we can insert a factor  $\psi_2$  between  $L^{-1}$  and  $L$  in the last term of (5.23), making an error that is in  $\mathfrak{S}_{(n-1)/(\frac{3}{2}-\varepsilon)}$ .

It remains to consider

$$\psi L^{-1} \psi_2 L_0 \psi_2 L_0^{-1} \psi_1 - \psi L^{-1} \psi_2 L \psi_2 L_0^{-1} \psi_1 = (\psi L^{-1} \psi_3)(\psi_2 L_0 \psi_2 - \psi_2 L \psi_2)(\psi_3 L_0^{-1} \psi_1),$$

where we have replaced  $\psi_2$  by  $\psi_2 \psi_3$ , with  $\psi_3 = 1$  on  $\text{supp } \psi_2$ , in a few places. Here the first factor is in  $\mathfrak{S}_{(n-1)/(\frac{1}{2}-\varepsilon)}$  by Lemma 5.10, the last factor is in  $\mathfrak{S}_{n-1}$  by Theorem 5.7, and the middle factor is a truncated  $\psi$ do of order zero, hence bounded in  $L_2$ , since  $P_{\gamma, \chi}$  and  $P_{0, \gamma, \nu}$  have the same principal symbol on  $V'$  (recall (4.6)). Then the whole expression is in  $\mathfrak{S}_{(n-1)/(\frac{3}{2}-\varepsilon)}$ . Thus we have obtained that  $\psi L^{-1} \psi_1 - \psi L_0^{-1} \psi_1 \in \mathfrak{S}_{(n-1)/(\frac{3}{2}-\varepsilon)}$ , which is contained in  $\mathfrak{S}_{n-1,0}$ . Together with (5.22) this shows

$$\psi L^{-1} \psi_1 - \psi(-P_{0, \nu, \gamma, +}) \psi_1 \in \mathfrak{S}_{n-1,0}.$$

Finally, since  $P_{0, \nu, \gamma}$  and  $P_{\nu, \gamma}$  have the same principal symbol (of order  $-1$ ) on  $V'$ ,  $\psi P_{0, \nu, \gamma, +} \psi_1 - \psi P_{\nu, \gamma, +} \psi_1$  is a truncated  $\psi$ do of order  $-2$ ; hence it is in  $\mathfrak{S}_{(n-1)/2} \subset \mathfrak{S}_{n-1,0}$ , and (5.21) follows.  $\square$

*Proof of Theorem 5.9.* We now consider  $L^{-1}$  on  $\Sigma_+$ , written as  $L^{-1} = \sum_{k=1}^N \varrho_k L^{-1}$  for some partition of unity  $\sum_{k=1}^N \varrho_k = 1$ . To analyze an individual term  $\varrho_k L^{-1}$ , we choose a cutoff function  $\psi_k$  that is 1 on  $\text{supp } \varrho_k$ , and write

$$\begin{aligned} (5.24) \quad \varrho_k L^{-1} &= \varrho_k L^{-1} \psi_k + \varrho_k L^{-1} (1 - \psi_k) = \varrho_k L^{-1} \psi_k + \varrho_k [L^{-1}, 1 - \psi_k] \\ &= \varrho_k L^{-1} \psi_k + \varrho_k L^{-1} [L, \psi_k] L^{-1}. \end{aligned}$$

Since  $[L, \psi_k]$  is a truncated zero-order  $\psi$ do, it is bounded in  $L_2$ . By Lemma 5.10,  $L^{-1} \in \mathfrak{S}_{(n-1)/(\frac{1}{2}-\varepsilon)}$ , so the last term satisfies

$$(5.25) \quad \varrho_k L^{-1} [L, \psi_k] L^{-1} \in \mathfrak{S}_{n-1+\delta}, \text{ any } \delta > 0.$$

We can assume that the supports of  $\varrho_k$  and  $\psi_k$  are so small that a diffeomorphism as described before Proposition 5.11 can be applied in a neighborhood of the supports; then Proposition 5.11 gives that

$$(5.26) \quad \varrho_k L^{-1} \psi_k = -\varrho_k P_{\nu, \gamma, +} \psi_k + R_{1,k}, \text{ with } R_{1,k} \in \mathfrak{S}_{n-1,0}.$$

A first observation resulting from this is that  $\varrho_k L^{-1} \psi_k$  is in  $\mathfrak{S}_{n-1}$ , since the  $\psi$ do of order  $-1$  is there. Then in view of (5.24)–(5.25),  $\varrho_k L^{-1} \in \mathfrak{S}_{n-1+\delta}$ , any  $\delta > 0$ . Summation in  $k$  gives that  $L^{-1} \in \mathfrak{S}_{n-1+\delta}$ . Next, we go back to (5.24), where the new information allows us to conclude that

$$\varrho_k L^{-1} [L, \psi_k] L^{-1} \in \mathfrak{S}_{(n-1+\delta)/2} \subset \mathfrak{S}_{n-1,0}.$$

In view of (5.26), we finally get that

$$\varrho_k L^{-1} = -\varrho_k P_{\nu, \gamma, +} \psi_k + R_{2,k}, \text{ with } R_{2,k} \in \mathfrak{S}_{n-1,0}.$$

Summation in  $k$  gives that

$$L^{-1} = -\sum_{k=1}^N \varrho_k P_{\nu, \gamma, +} \psi_k + \sum_{k=1}^N R_{2,k} = -P_{\nu, \gamma, +} + R_3,$$

with  $R_3 \in \mathfrak{S}_{n-1,0}$ .  $\square$



#### 5.4 Reduction of the Poisson operators.

We now consider the operator (5.1). To find the spectral behavior, we note that by the general rule for eigenvalues  $\mu_j(ST) = \mu_j(TS)$ , we can write

$$(5.27) \quad \mu_j(i_V \gamma_V^{-1} L^{-1} (\gamma_V^{-1})^* \text{pr}_V) = \mu_j(L^{-1} (\gamma_V^{-1})^* \text{pr}_V i_V \gamma_V^{-1}) = \mu_j(L^{-1} (\gamma_V^{-1})^* \gamma_V^{-1}),$$

in view of (2.15).

**Lemma 5.12.** *The operator  $(\gamma_V^{-1})^* \gamma_V^{-1}$  satisfies*

$$(5.28) \quad (\gamma_V^{-1})^* \gamma_V^{-1} = P_{1,+},$$

where  $P_1 = K_\gamma^* K_\gamma$  is a selfadjoint nonnegative elliptic  $\psi$ do of order  $-1$  on  $\Sigma$  with principal symbol  $(2|\xi'|)^{-1}$ .

*Proof.* We have for  $\varphi, \psi \in X$ :

$$(5.29) \quad ((\gamma_V^{-1})^* \gamma_V^{-1} \varphi, \psi)_{X^*, X} = (\gamma_V^{-1} \varphi, \gamma_V^{-1} \psi)_V = (K_\gamma \varphi, K_\gamma \psi)_H \\ = (K_\gamma^* K_\gamma \varphi, \psi)_{\frac{1}{2}, -\frac{1}{2}} = (P_1 \varphi, \psi)_{\frac{1}{2}, -\frac{1}{2}},$$

where  $P_1 = K_\gamma^* K_\gamma$  is a  $\psi$ do of order  $-1$  on  $\Sigma$ , by the rules of calculus for pseudodifferential boundary operators; it is clearly selfadjoint nonnegative. The principal symbol is found from the calculation using the principal symbol-kernel  $\tilde{k}^0 = e^{-x_n |\xi'|}$  of  $K_\gamma$ :

$$\int_0^\infty e^{-x_n |\xi'|} e^{-x_n |\xi'|} dx_n = (2|\xi'|)^{-1},$$

also equal to  $\|\tilde{k}^0\|_{L_2(\mathbb{R}_+)}^2$ . Since  $\varphi$  and  $\psi$  are supported in  $\Sigma_+$ , (5.29) may be rewritten further as

$$(P_1 \varphi, \psi)_{\frac{1}{2}, -\frac{1}{2}} = (P_1 e^+ \varphi, e^+ \psi)_{\frac{1}{2}, -\frac{1}{2}} = (r^+ P_1 e^+ \varphi, \psi)_{H^{\frac{1}{2}}(\Sigma_+^*), H_0^{-\frac{1}{2}}(\Sigma_+)} = (P_{1,+} \varphi, \psi)_{X^*, X}.$$

Then (5.28) follows since  $\varphi$  and  $\psi$  are arbitrary.  $\square$

Next, we define

$$(5.30) \quad P_2 = P_1^{\frac{1}{2}},$$

a nonnegative selfadjoint  $\psi$ do on  $\Sigma$  of order  $-\frac{1}{2}$ , by Seeley [S67]. Moreover, set

$$(5.31) \quad G^{(1)} = P_{1,+} - (P_{2,+})^2, \quad G^{(\frac{1}{2})} = (P_{1,+})^{\frac{1}{2}} - P_{2,+}.$$

**Lemma 5.13.** *When  $n \geq 3$ ,  $G^{(1)} \in \mathfrak{S}_{n-2}$  and  $G^{(\frac{1}{2})} \in \mathfrak{S}_{2(n-2)}$ . When  $n = 2$ ,  $G^{(1)} \in \mathfrak{S}_{n-1,0}$  and  $G^{(\frac{1}{2})} \in \mathfrak{S}_{2(n-1),0}$ .*

*Proof.* We first note (cf. (5.12)) that

$$(5.32) \quad G^{(1)} = P_{1,+} - P_{2,+} P_{2,+} = r^+ P_2 P_2 e^+ - r^+ P_2 e^+ r^+ P_2 e^+ = L(P_2, P_2).$$

Since  $P_2$  is a  $\psi$ do of order  $-\frac{1}{2}$ , we have by Theorem 5.2 that  $L(P_2, P_2)$  is in  $\mathfrak{S}_{n-2}$  when  $n \geq 3$ . For  $n = 2$ , we see that  $s_j(L(P_2, P_2)) j^{1/(n-1)} \rightarrow 0$  for  $j \rightarrow \infty$  by use of Remark 5.8.

To obtain the result for  $G^{(\frac{1}{2})}$ , we shall as in [G83] appeal to a result of Birman, Koplienko and Solomyak [BKS75]. It states that when  $M_1$  and  $M_2$  are compact selfadjoint nonnegative operators on a Hilbert space  $H$  such that  $G^{(1)} = M_1 - M_2$  is in  $\mathfrak{S}_\gamma$  for some  $\gamma > 0$ , then  $G^{(\sigma)} = M_1^\sigma - M_2^\sigma$  is in  $\mathfrak{S}_{\gamma/\sigma}$  for all  $0 < \sigma < 1$ . Applying this with  $M_1 = P_{1,+}$ ,  $M_2 = (P_{2,+})^2$  and  $\sigma = \frac{1}{2}$ , we get the desired result when  $n \geq 3$ . The paper [BKS75] also shows that  $\limsup s_j(G^{(\frac{1}{2})}) j^{1/(2n-2)}$  is dominated by  $\limsup s_j(G^{(1)}) j^{1/(n-1)}$ , which assures the statement for  $n = 2$ .  $\square$

Now we continue the analysis in (5.27) as follows:

**Proposition 5.14.**

$$(5.33) \quad \mu_j(\mathrm{i}_V \gamma_V^{-1} L^{-1} (\gamma_V^{-1})^* \mathrm{pr}_V) = \mu_j(P_{2,+} L^{-1} P_{2,+} + G'),$$

where  $G'$  is the selfadjoint operator

$$(5.34) \quad G' = G^{(\frac{1}{2})} L^{-1} P_{2,+} + P_{2,+} L^{-1} G^{(\frac{1}{2})} + G^{(\frac{1}{2})} L^{-1} G^{(\frac{1}{2})};$$

it is in  $\mathfrak{S}_{(n-1)/2-r}$  for a positive  $r$  when  $n \geq 3$ , and in  $\mathfrak{S}_{(n-1)/2,0}$  when  $n = 2$ .

*Proof.* Using Lemma 5.13 and (5.27) and the definitions (5.30)–(5.31) we have:

$$\begin{aligned} \mu_j(\mathrm{i}_V \gamma_V^{-1} L^{-1} (\gamma_V^{-1})^* \mathrm{pr}_V) &= \mu_j(L^{-1} P_{1,+}) = \mu_j(L^{-1} (P_{1,+})^{\frac{1}{2}} (P_{1,+})^{\frac{1}{2}}) \\ &= \mu_j((P_{1,+})^{\frac{1}{2}} L^{-1} (P_{1,+})^{\frac{1}{2}}) = \mu_j(P_{2,+} L^{-1} P_{2,+} + G'), \end{aligned}$$

where  $G'$  is as in (5.34). When  $n \geq 3$ , we use that  $L^{-1} \in \mathfrak{S}_{n-1}$ ,  $P_{2,+} \in \mathfrak{S}_{2(n-1)}$ , and  $G^{(\frac{1}{2})} \in \mathfrak{S}_{2(n-2)}$  (by Lemma 5.13) and the rule (5.2) to see that

$$G' \in \mathfrak{S}_p \text{ with } p = \left( \frac{1}{n-1} + \frac{1}{2(n-1)} + \frac{1}{2(n-2)} \right)^{-1} < \frac{n-1}{2},$$

hence  $G' \in \mathfrak{S}_{(n-1)/2-r}$  for a positive  $r$ . When  $n = 2$ ,  $G^{(\frac{1}{2})} \in \mathfrak{S}_{2(n-1),0}$  leads to  $G' \in \mathfrak{S}_{(n-1)/2,0}$ .  $\square$

We can then conclude:

**Theorem 5.15.** *The eigenvalues of  $A_{\chi, \Sigma_+}^{-1} - A_\gamma^{-1}$  satisfy:*

$$(5.35) \quad \mu_j(A_{\chi, \Sigma_+}^{-1} - A_\gamma^{-1}) = \mu_j(\mathrm{i}_V \gamma_V^{-1} L^{-1} (\gamma_V^{-1})^* \mathrm{pr}_V) = \mu_j(P_{2,+} P_{\nu, \gamma, +} P_{2,+} + G),$$

where  $G \in \mathfrak{S}_{(n-1)/2,0}$ .

*Proof.* This follows by inserting the information from Theorem 5.9 in the formula (5.33), using that  $P_{2,+} R P_{2,+} \in \mathfrak{S}_{(n-1)/2,0}$  by the rules in Section 5.1.  $\square$

**5.5 Spectral asymptotics.**

To find the asymptotic behavior of the  $s$ -numbers we shall use the following theorem shown in [G11a] (Th. 3.3):

**Theorem 5.16.** [G11a] *Let  $P$  be an operator on  $\Sigma$  composed of  $l$  classical pseudodifferential operators  $P_1, \dots, P_l$  of negative orders  $-t_1, \dots, -t_l$  and  $l+1$  functions  $b_1, \dots, b_{l+1}$  that are piecewise continuous on  $\Sigma$  with possible jumps at  $\partial \Sigma_+$*

$$(5.36) \quad P = b_1 P_1 \dots b_l P_l b_{l+1}.$$

Let  $t = t_1 + \dots + t_l$ . Then  $P$  has the spectral behavior:

$$(5.37) \quad s_j(P) j^{t/(n-1)} \rightarrow c(P) j^{t/(n-1)} \text{ for } j \rightarrow \infty,$$

where

$$(5.38) \quad c(P) = \frac{1}{(n-1)(2\pi)^{(n-1)}} \int_{\Sigma} \int_{|\xi'|=1} |b_1 \dots b_{l+1} p_1^0 \dots p_l^0|^{(n-1)/t} d\omega(\xi') dx'.$$

Let us also recall that the principal symbol of  $P_{\nu, \gamma}$  is  $p^0 = -|\xi'|^{-1}$ . As noted in Lemma 5.12, the principal symbol of  $P_1 = K_\gamma^* K_\gamma$  is  $\|\tilde{k}^0\|_{L_2}^2 = (2|\xi'|)^{-1}$ ; that of the squareroot  $P_2$  is  $\|\tilde{k}^0\|_{L_2} = (2|\xi'|)^{-\frac{1}{2}}$ .

Then we can finally show:

**Theorem 5.17.** *Let  $\lambda \in \varrho(A_{\chi, \Sigma_+}) \cap \varrho(A_\gamma)$ . The  $s$ -numbers of  $(A_{\chi, \Sigma_+} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1}$  satisfy the asymptotic formula*

$$(5.39) \quad s_j((A_{\chi, \Sigma_+} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1})j^{2/(n-1)} \rightarrow C_{0,+}^{2/(n-1)} \text{ for } j \rightarrow \infty,$$

where

$$(5.40) \quad C_{0,+} = \frac{1}{(n-1)(2\pi)^{n-1}} \int_{\Sigma_+} \int_{|\xi'|=1} (\|\tilde{k}^0\|_{L_2(\mathbb{R}_+)} |p^0|^{1/2})^{n-1} d\omega(\xi') dx' = c_n \int_{\Sigma_+} 1 dx'.$$

for a constant  $c_n$  depending on  $n$  (see (5.41) below).

*Proof.* We first treat the case without  $\lambda$  (or with  $\lambda = 0$ ), where the realizations are positive. Here the  $s$ -numbers are the positive eigenvalues, and we use (5.35). We can identify  $P_{2,+}P_{\nu,\gamma,+}P_{2,+}$  with the operator  $1_{\Sigma_+}P_21_{\Sigma_+}P_{\nu,\gamma}1_{\Sigma_+}P_21_{\Sigma_+}$  in  $L_2(\Sigma)$ , acting trivially (as 0) on  $L_2(\Sigma_-)$ . An application of Theorem 5.16 to this operator gives that

$$\mu_j(1_{\Sigma_+}P_21_{\Sigma_+}P_{\nu,\gamma}1_{\Sigma_+}P_21_{\Sigma_+})j^{2/(n-1)} \rightarrow c^{2/(n-1)} \text{ for } j \rightarrow \infty,$$

where

$$\begin{aligned} c &= \frac{1}{(n-1)(2\pi)^{(n-1)}} \int_{\Sigma} \int_{|\xi'|=1} |1_{\Sigma_+}p_2^0p_2^0|^{(n-1)/2} d\omega(\xi') dx' \\ &= \frac{1}{(n-1)(2\pi)^{(n-1)}} \int_{\Sigma_+} \int_{|\xi'|=1} (\|\tilde{k}^0\|^2 |p^0|)^{(n-1)/2} d\omega(\xi') dx' = C_{0,+}. \end{aligned}$$

Since  $G \in \mathfrak{S}_{(n-1)/2,0}$ , this asymptotic behavior is preserved under addition of  $G$ , by Lemma 5.1 1°, which implies the main statement in the theorem for  $\lambda = 0$ .

Since  $\|\tilde{k}^0\|^2 = (2|\xi'|)^{-1}$ ,  $|p^0| = |\xi'|^{-1}$ , the constant  $c_n$  can be calculated as

$$\begin{aligned} (5.41) \quad c_n &= \frac{1}{(n-1)(2\pi)^{(n-1)}} \int_{|\xi'|=1} 2^{-(n-1)/2} d\omega(\xi') \\ &= \frac{1}{(n-1)(2\pi)^{(n-1)}} 2^{-(n-1)/2} (n-1) \pi^{(n-1)/2} \Gamma(1 + \frac{n-1}{2})^{-1} \\ &= (2\pi)^{-(n-1)/2} 2^{1-n} \Gamma(1 + \frac{n-1}{2})^{-1}. \end{aligned}$$

For more general  $\lambda \in \varrho(A_{\chi, \Sigma_+}) \cap \varrho(A_\gamma)$ , we use a resolvent identity as in [G11a]:

$$\begin{aligned} (5.42) \quad (B - \lambda)^{-1} - (B_1 - \lambda)^{-1} &= (1 + \lambda(B_1 - \lambda)^{-1})(B^{-1} - B_1^{-1})(1 + \lambda(B - \lambda)^{-1}) \\ &= B^{-1} - B_1^{-1} + \lambda(B_1 - \lambda)^{-1}(B^{-1} - B_1^{-1}) + (B^{-1} - B_1^{-1})\lambda(B - \lambda)^{-1} \\ &\quad + \lambda(B_1 - \lambda)^{-1}(B^{-1} - B_1^{-1})\lambda(B - \lambda)^{-1}, \end{aligned}$$

valid for  $\lambda, 0 \in \varrho(B) \cap \varrho(B_1)$ . We apply it to  $B = A_{\chi, \Sigma_+}$  and  $B_1 = A_\gamma$  for  $\lambda \in \varrho(\tilde{A}) \cap \varrho(A_\gamma)$ . Since  $(A_\gamma - \lambda)^{-1}$  and  $(A_{\chi, \Sigma_+} - \lambda)^{-1}$  are in  $\mathfrak{S}_{n/2}$  (cf. Corollary 3.2), the three last terms are in  $\mathfrak{S}_{(n-1)/2-r}$  with  $r > 0$ . Then we find by Lemma 5.1 1° that the main asymptotic estimate of the  $s$ -numbers is the same as for  $A_{\chi, \Sigma_+}^{-1} - A_\gamma^{-1}$ .  $\square$

For  $n \geq 3$ , a generalization of Laptev's result in Theorem 5.2 to nonstandard  $\psi$ do's like  $\Lambda_+$  and  $\Lambda_-$  would allow an estimate of  $s_j(A_{\chi, \Sigma_+}^{-1} - A_\gamma^{-1}) - C_{0,+}^{2/(n-1)}j^{-2/(n-1)}$  by a lower power of  $j$ .

The methods of [G11] would be useful in an extension of the results to exterior domains.

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